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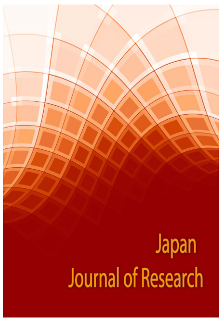

A Duality to the Birch & Swinnerton-Dyer Conjecture and the Visualized Sandwich Proof to the Riemann Hypothesis

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Abstract

The truthfulness of the Riemann Hypothesis has struggled for a hundred of years since the 19th Century in the year 1859. However, until now, even the most mathematicians still cannot make a proof or disproof for such a hypothesis that is convincing for everyone. The most difficult part in the proof or disproof of the hypothesis is the issue of how one may determine if there are no other critical lines like x = 0.5 that contain the non-trivial zeta zeros in the Critical Strip Region 0 < x < 1. This writer first employs the method of "Proving by a Contradiction" and assume that rather than Z = Re(ξ(s = 0.5 + Iy)) = 0, there is also an additional Z = Re(ξ(s" = x" + Iy)) = 0. However, with the help of a computer software named "Mathematica" program segment code:

"Plot[Evaluate[ReIm[Zeta[0.5 + I t]]], {t, 0, 30}]" which is just the line

Z = Re(ξ(s")) = 0, without any other line equal to zero. Obviously, Z = Re(ξ(s = x + Iy)) = 0 at x = 0.5 leads to a contradiction immediately with the assumption that there should be another x" not equal to 0.5 but Z = Re(ξ(s")) = 0. This result implies that x = 0.5 is the only critical line at x = 0.5. Moreover, for all the other roots of, this writer has found an ε-δ relationship between Z = Re(ξ(s+/-δ)) = +/-εZ where s = x + Iy which is sandwiched for a convergence to the Z = 0. There is also another proof by contradiction to show that there is one and only one critical line for x = 0.5. Therefore, both of these contradictory proofs show that there is only one critical line at x = 0.5, implying that the Riemann Hypothesis must be true or the critical line x = 0.5 must contain all of the non-trivial Riemann Zeta zeros.

Introduction

In the present paper, the author attempts to provide two proofs for the Riemann Hypothesis with the help of computer-simulated program segments and the associated graphics with the Sandwiched Theorem. In addition, the author has found a duality for the Birch and Swinnerton-Dyer Conjecture. In fact, all three proofs are interesting..

Computational Results

1. Visualized Sandwich Critical Line (which is a visualization of the zeta function)

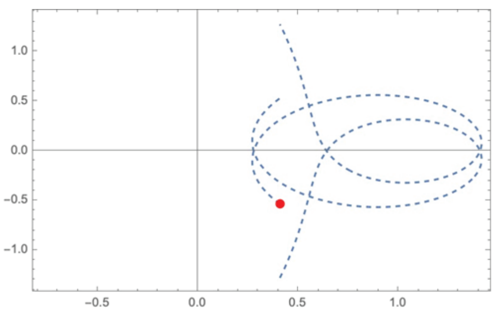

Figure Zeta ((1/1.1) + Iy)

```
tab1 = Table[{1/1.1, j}, {j, -15, 15, 0.1}];
tab2 = Table[ReIm[Zeta[1/1.1 + j I]], {j, -15, 15, 0.1}];

lp1 = ListPlot[tab1, Joined -> True, PlotStyle -> Dashed,
   Epilog -> {Red, PointSize[0.02], Point /@ tab1},
   Frame -> True, ImageSize -> 400];

lp2 = ListPlot[tab2, Joined -> True, PlotStyle -> Dashed,
   Epilog -> {Red, PointSize[0.02], Point /@ tab2},
   Frame -> True, ImageSize -> 400];

an = Row /@ Thread[{lp1, lp2}]
```

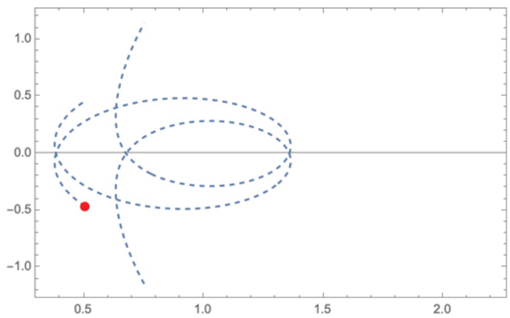

Figure Zeta ((1/0.9) + Iy)

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```
tab1 = Table[{1/1, j}, {j, -15, 15, 0.1}];
tab2 = Table[ReIm[Zeta[1/1 + j I]], {j, -15, 15, 0.1}];

lp1 = ListPlot[tab1, Joined -> True, PlotStyle -> Dashed,
       Epilog -> {Red, PointSize[0.02], Point[#]}, Frame -> True,
       ImageSize -> 400] & /@ tab1;

lp2 = ListPlot[tab2, Joined -> True, PlotStyle -> Dashed,
    Epilog -> {Red, PointSize[0.02], Point[#]}, Frame -> True,
    ImageSize -> 400] & /@ tab2;

an = Row /@ Thread[{lp1, lp2}]
```

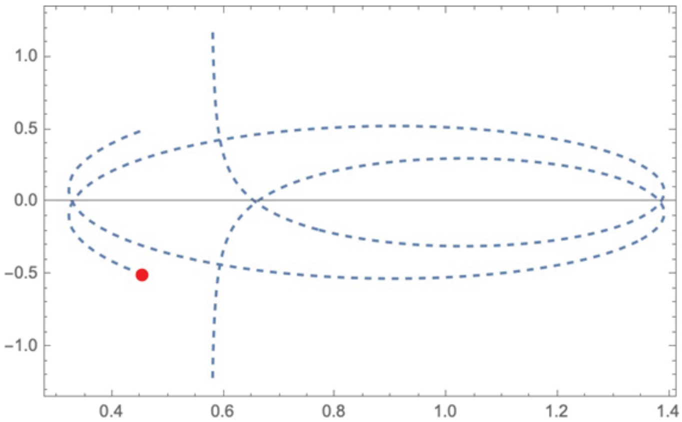


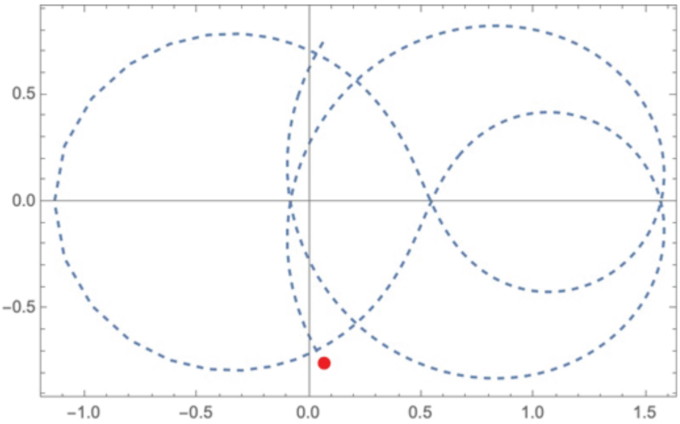
Figure Zeta ((1/1) + Iy)

Suppose there is another critical line with a real part NOT equal to 0.5 and some imaginary parts that constitute the non-trivial zeros [8] of the Riemann hypothesis. Then [8], there should be a left-hand 1/ (x - 0.1) and right-hand limit 1/(x + 0.1) for x, where x is not equal to 0.5, which tends to the squeezing limit of zero. Now, consider the case for ξ(s) when s = 1/n = 1/0.9, 1/n = 1/1, and 1/n = 1/1.1, and by the above Mathematica codes, we may obtain the above left-hand and right-hand limits, which are tending to the same direction but NOT squeezing from left and right towards zeros as the case for 1/1.9 or the left-hand limit and 1/2.1 + jI or the right-hand limit which is squeezing 1/2 + jI to approach 0. Hence, our assumption that 1/1 is another critical line, similar to the case of 1/2, such that 1/1.1 and 1/0.9 will squeeze the 1/1 approach to zero, is incorrect, as shown in the visualized figures above. Thus, 1/1 must NOT be a non-trivial zeta zero (which must have the property that the left-hand and right-hand limits squeeze it to approach zero). Hence, there is a contradiction to the initial assumption that all non-trivial zeta zeros must have their left-hand and right-hand limits for a squeezing approach to a zero. In other words, the assumption that 1/1 is a non-trivial zeta zero must be wrong as it does not possess the squeezing to zeros from the left and right limit's property. The lack of such a property implies that 1/1 should be another type of zero instead of non-trivial zeta zeros.

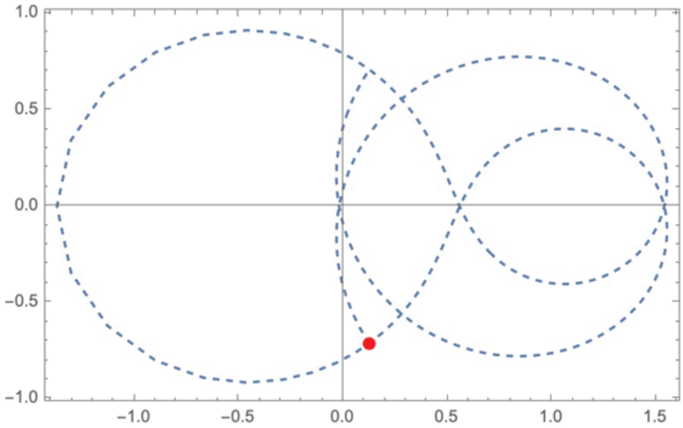
As we know, all negative even numbers are just the trivial zeros of the Riemann Zeta function but NOT the non-trivial zeros.

Therefore, this writer concludes that 1/1 must be a pole with a sudden drop rather than a critical line of non-trivial zeros like the case of 1/2 or an ordinary zero for the Riemann Zeta function. Thus, the Riemann Hypothesis cannot be false for 1/1 (a pole but NOT a non-trivial zero) in such a case, as proved in [2].

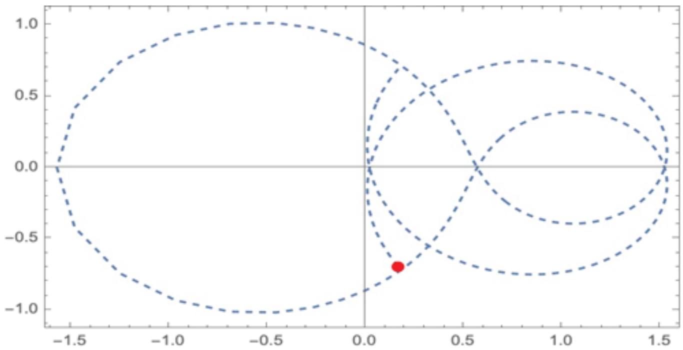
Zeta (1/2.5) — left hand limit



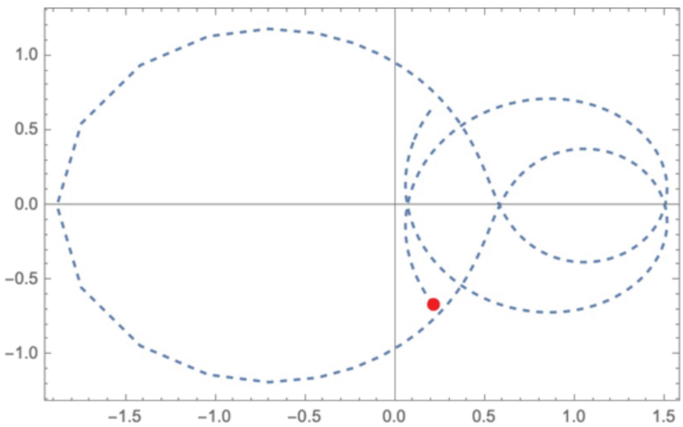
Zeta (1/2.1) — left hand limit



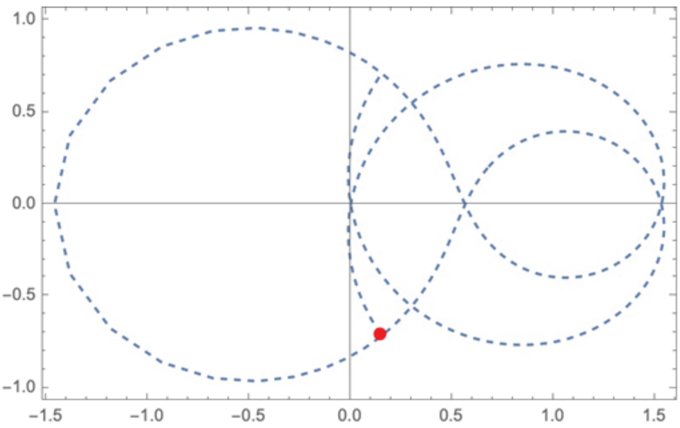
Zeta (1/1.9) — Right hand limit



Zeta (1/1.7) — Right hand limit



Zeta (1/2) was sandwiched to zero



(N.B. The above figures which sandwiched the Zeta Function ξ(s) = 0 from the left-hand limit and the right limit in the Critical Strip Region as 1/2.5 = 0.4 and

1/1.7 = 0.588 are actually different from what the trivial zero -- s = 0 is, that is the present testing s is staying in the suspected non-trivial zeros (or the so-called Critical Strip) Region which means that s in fact stays between 0 and 1 or in the critical strip region, 0 < s < 1, but NOT those trivial zeros that stay outside the region s < 0 and s > 1 which implies the results that ξ(-2) = ξ(-4) = ξ(-6) =…..= 0. In practice, the "0" in ξ(0) = -12 which is different from the present "0" for the answer in the ξ(s = 0.5 + Iy) = 0, which is sandwiched by ξ(s = 1/2.5 = 0.4 + Iy) and ξ(s = 1/1.7 = 0.588 + Iy). In fact, the general trend of the above outcome (both the 0.4 < s < 0.588 and their computer-simulated figures (results) are sandwiched from left to right) is definitely different from what we obtained previously in the first section's general trend -- ξ(s = 1/0.9 = 1.111 + Iy) and ξ(s = 1/1.1 = 0.9090 + Iy) which also sandwich the pole 1/1.1 < 1/1 < 1/0.9 from left to right but the computer-simulated figures (or the results) are just moving in the same direction. If we compare the above two cases, they are actually different, which implies that s = 0.5 + Iy is the non-trivial zeta zeros line or the critical line while s = 1 + Iy is the pole in the Critical Strip Region for 0 < s < 1. In practice, if we consider the tendency of the approaching, the line like the x = will move from right like the line x = 1/1.1 to the left until a stop at the x = 1/2 which is equal to a zero, that is, right-handed critical strip region 0.5 < x < 1 with x≠1. Then, the tendency turns around from the left x = 1/2.5 to the right x = 1/1.7. The above phenomena for such a sandwiched limit tendency from the left and right will continue because of the well-known analytic continuation property of the zeta function. The analytic continuation has been proven to be true for the Riemann Zeta function (except for the line at x = 1, or known as the pole) in the whole complex plane, or in particular for the left-handed critical strip region 0 < x < 0.5 with x≠0.5. Hence, there should be infinite equivalent lines or points in the Critical Strip Region, 0 < x < 1 where x≠1 like the Re(s) = 0.5 (or known as the critical line which contains the non-trivial zeta zeros) and Re(s) = 1 (which is known as the pole) because of the analytic continuation property. However, we have never found any other lines, such as the critical line x = 0.5 or x = 1 or even a point from the present computer simulation, together with the left- and right-sandwiched limiting tendency for the whole critical strip region 0 < x < 1. This may contradict the assumption that there should be another critical line besides Re(s) = 0.5 and Re(s) = 1. Thus, there must be only one critical line at Re(s) = 0.5.

Even if we suppose for every point or every line s1 = x1 + Iy such that it is outside the critical line x = 0.5 + yI but remains in the critical strip region 0 < x < 1 such that ξ(s1) = 0. Then, by Rolle's Theorem of the Mathematical Real Analysis, there is a constant "c" between s1 and 0.5 such that ξ'(c) = 0. However, the classical zeta function does NOT converge in the critical strip region 0 < x < 1. The only way is to consider a convergent function, as follows [3, p.176]:

$$\xi(s) = 1 - \frac{1}{1-s} - s\int_1^\infty \left(\frac{\{t\}}{t^{1+s}}\right) dt$$

where the integral $s\int_1^\infty \left(\frac{\{t\}}{t^{1+s}}\right) dt$ is an analytic function in the region.

Now, differentiating ξ(s) with respect to s, we obtain

$$\xi'(s) = (-1)\ (1-s)^{-2} - \left[s\int_1^\infty \left(\frac{\{t\}}{t^{1+s}}\right) dt + s\int_1^\infty \left((-1-s)e^{(-1-s)ln(t)}\right) dt \right]$$

$$= \frac{-1}{(1-s)^2} - \left[\frac{-1}{s-1} + \frac{(-1-s)}{s}\right]$$

$$= 0$$

Hence, $s = \frac{\left(-1 \pm \sqrt{5}\right)}{2}$ i.e. c = 0.6180339887 or c = -1.6180339887 (rejected as c < 0)

According to the Mathematical Real Analysis -- Rolle's Theorem, there is only one point "c" that stays between s1 and 0.5 such that ξ(s1) = 0, ξ(0.5) = 0 with ξ'(c) = 0. Or s1 = 0.7360679774. In practice, ξ(s1 = 0.7360679774 + Iy) will only infinitely approach zero with a small ε but never meet at zero or diverge (possibility there may be an impulse at s1 = 0.7360679774). That is, s1 = 0.7360679774 + Iy, which is supposed to be a feasible non-trivial zeta zero outside the critical line x = 0.5, is just giving us a false expectation. The above contradiction implies either something wrong with Rolle's theorem or the assumption about the existence of another zeta non-trivial root, 0 < s1 < 0.5 or 0.5 < s1 < 1, may be INCORRECT. As Rolle's theorem has been verified for hundreds of years, the only possible reason for the contradiction must be the wrong assumption in the existence of another non-trivial zeta zeros s1 outside the critical line x = 0.5. Therefore, we can conclude that there must be no other Zeta non-trivial roots outside the critical line x = 0.5.

(N.B. The author has already verified the feasible s1 = 0.7360679774 + Iy non-trivial zeta zero candidate by the U.S.A. The MatLab contour integration program code in [4] produces no positive sharp results, just like the case in x = 0.5:

(-1.2454 – 7.8232i) with the norm much greater than zero Vs x = 0.736: (1.9595e-14 + 5.7732e-15i) with the norm approaching 0 between the values of 13.75 and 14.25. In addition, there is also another feasible non-trivial Zeta zeros candidate as the s2 = 0.2639320226 which has also been proved by this writer in the similar way to be a false root for the Zeta function.) The figure below shows the coordinated-geometric distribution of these false roots as the critical line x = 0.5 in the next page.

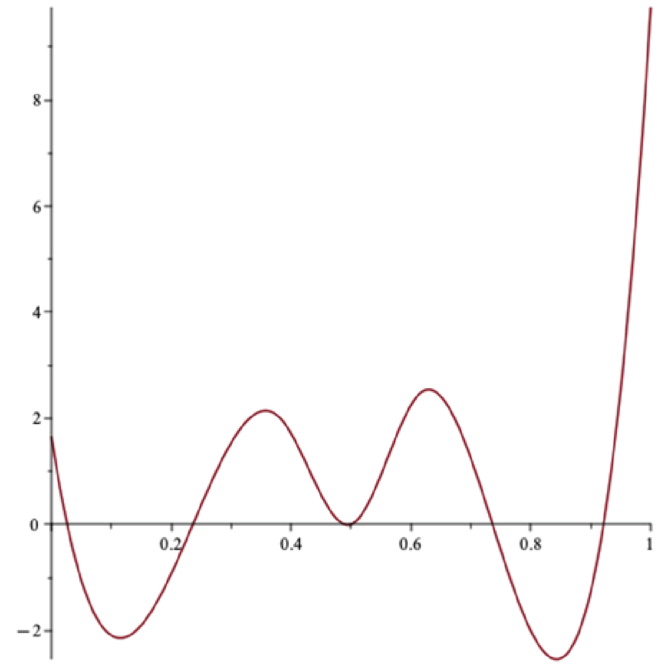


Figure 0: A proposed Coordinated-Geometric Distributed of Both False Roots & Critical Line (or Roots) in the Critical Strip Region (which is generated by the Maple Soft 2024) or the "Internal Structure of the Critical Strip Region". In a similar way, we may substitute back those values like $\zeta(2) = \pi^2/6$ etc for the interpolating a polynomial just similar to the above one and plot as much range of the zeta function as possible in order to investigate the full structure of the Riemann Zeta function. In practice, there may be a sharped impulse induced which sandwiches by the left hand and the right of the upward maximum when they tends to a zero. The outcome may be the commercial engineering type of the theoretical perfect (or ideal) filter or control system (may be worked with the perfect (or ideal feed-back mechanism) as the reference for the practical digital signal processing to produce the sharpest of the sharpest image etc. (N.B. There is NO perfect filter with perfect feed-back system in the real world.) Actually, in the future research, we may focus in the paired upward and downward impulse(s) for those peak(s) and this author will leave to the interested parties in the field of commercial engineering for a continuation with some depth and wide investigations in making a better control system such as in the case of our present everyday used Hi-Fi.

(N.B. Certainly in the vice versa (or mirror image) way, we may also have the commercial reverse filter & feedback system for the inverse control system etc. In fact, one may consider the program coding in the conclusion section for some more details in the recursive issues. In fact, the optimized regularized sum of the Riemann Zeta Function $\sum_{n=1}^{\infty}\frac{1}{n^s}$ may be equal to zero as the above curve just touched the x-axis with repeated roots. One interesting phenomenon is that one may consider the interpolated curve cuts the x-axis as the roots of $\zeta'(s)$ but at the same time, these roots are just the false roots of $\zeta(s)$. This is because the differentiation of the maxima/minima of a curve are just the roots of the curve's own derivative function. Or the integration of the roots from the curve's own derivative function will give you back the maxima/minima of the original curve. By repeating the above process infinity, we may find a multiple layers of layers' tower of integration or differentiation which progress an "if and only if" of roots and maxima/minima relationship with one another. In the vice versa (or the mirror inverse way, given the tower of the multiple layers of layers tower of integration or differentiation relationship, we may get back the all of the values of roots and maxima/minima of the curves.)

A Duality to the Birch & Swinnerton-Dyer Conjecture

Using the standard logarithmic inequality [7] $ln(1+x) \leqslant x$.

$$log\left(\frac{N_p}{p}ln\left(\frac{N_p}{p}\right)-\frac{N_p}{p}+1\right) \leqslant \frac{N_p}{p}log\left(ln\left(\frac{N_p}{p}\right)-\frac{N_p}{p}\right)= \frac{N_p}{p}log\left[ln\left(\frac{N_p}{p}\right)-1\right]$$

But for a logarithm, it must greater than or equal to zero, thus,.

$$-\frac{N_p}{p}log\left(ln\left(\frac{N_p}{p}\right)\right) \leqslant log\left(\frac{N_p}{p}ln\left(\frac{N_p}{p}\right)-\frac{N_p}{p}+1\right) \leqslant \frac{N_p}{p}log\left(ln\left(\frac{N_p}{p}\right)\right)$$

i.e.

$$\left|\left(log\left(\frac{N_p}{p}ln\left(\frac{N_p}{p}\right)-\frac{N_p}{p}+1\right)-\frac{N_p}{p}log\left(ln\left(\frac{N_p}{p}\right)\right)\right)\right| \leqslant logC$$

or

$$\left|\left(log\left(\prod_{p\leqslant x}\frac{N_p}{p}\right)-\frac{N_p}{p}log\left(ln\left(\frac{N_p}{p}\right)\right)\right)\right| \leqslant logC$$

$$\left|\left(log\left(\prod_{p\leqslant x}\frac{N_p}{p}\right)-xlog\left(ln(x)\right)\right)\right| \leqslant logC'$$

which is, in fact, the radius of convergence for the function of.

(N.B. Similarly, we may replace another part of the logarithmic function by x, but the remaining computation are in fact the same, this writer will NOT repeat.)

$$\left|\left(log\left(\prod_{p\leqslant x}\frac{N_p}{p}\right)\right)\right| \leqslant logC+\frac{N_p}{p}log\left(ln(x)\right) \leqslant logC+rlog\left(ln(x)\right)$$

$$\left(\prod_{p\leqslant x}\frac{N_p}{p}\right) = C*ln(x)^r$$

$= C*ln(x)^r$

$= C*ln(x)^r$ for $x \to \infty$ (by the Intermediate Value Theorem)

$$\lim_{x\to\infty}\left(\prod_{p\leqslant x}\frac{N_p}{p}\right)=C*ln(x)^r$$

In fact, what the other part of the duality is just:

$$\left(\prod_{x\leqslant p}x\right)=C*ln\left(\frac{N_p}{p}\right)^r$$

Discussion: A Novel Views For the Further Riemann Hypothesis Research – How We May Determine the Critical Line Is Optimized or the best trajectory

On the contrary, assume that there are another critical line s' in the critical region $0 < \Re(s') \neq 0.5 < 1$ such that $\xi(Re(s') + Iy) = 0$ which has all of the same properties as $\xi(0.5 + Iy) = 0$. In practice, let s' = Re (s') + Iy. Then there must be a line like x' = Re(s') such that the line x' must contain the point s' = Re(s') + Iy. (N.B. In the present proof, we authors use Re(s') = 0.1 or Re(s') = 0.7 as the case studies in the following discussion.) Consider the Critical strip region, for every x' between 0 and 1 (or $0 < x' < 1$) as well as for every given

$$\varepsilon_Z \pm I\,\varepsilon_y = d\Re\left(\zeta\left(x'\pm\delta_x \pm Iy\pm\delta_y\right)\right), \Re\left(\zeta\left(\Re(s')\pm\delta_x\pm Iy\right)\right)$$

there is an existing

$$\left(\delta_x \pm I\,\delta_y\right) = \zeta^{-1}\left(\pm\varepsilon_Z + \zeta\left(x' - \Re\left(s'\right)\right)\right) - \left(\pm\varepsilon_y + \left(Iy - Iy\right)\right)$$

such that δ_x will approach to $|(x'-\Re(s'))|$ and δ_y will approach to |(y - y)| when the (ε_Z +/- I ε_y) tends to zero. Topologically, the above mathematical analysis statement implies that, in general, we may consider a 3-dimensional complex open sphere, given any

$(\varepsilon_Z, \varepsilon_y) = d\,(\Re\,(\zeta\,(x' \pm \delta_x \pm Iy \pm \delta_y)), \Re\,(\zeta\,(\Re\,(s') \pm \delta_x \pm Iy)))$,

there will always be a (δ_x, δ_y) with

$(\delta_x, \delta_y) = (\Re\,(\zeta^{-1}\,(\pm\,\varepsilon_Z + \Re\,(\zeta\,(x' - \Re\,(s') \pm Iy))), (\pm\,\varepsilon_y + (Iy - Iy))))$

where $(\delta_x, \delta_y) \rightarrow (|\,(x' - \Re\,(s'))|, |(y - y)|)$ when $(\varepsilon_Z, \varepsilon_y) \rightarrow (0,0)$.

(N.B. d is a metric in the mathematical language of the point set topology.)

Therefore, by the Limit Squeezing Principle or Sandwich Theorem, all of the other zeros (real part meets the imaginary part plus the line $Z = \varepsilon_Z$) must stay outside the critical line of x' = s'

when the geometric distance δ_x approaches $|(x' - s')|$ and ε_x also tends to zero from both sides of the upper limiting distance (downwards along the y-axis to zero) or the

$$inf\,\{0 < \varepsilon_Z < 1 \mid d\left(\Re\left(\zeta\left(x' - \delta_x\right) \pm Iy\right), \Re\left(\zeta\left(\Re\left(s'\right) \pm Iy\right)\right)\right) < \varepsilon_Z\}$$

and the lower limiting distance (upwards along the x-axis to zero) or the

$$sup\,\{0 < -\varepsilon_Z < 1 \mid d \left(\Re\left(\zeta\left(x' - \delta_x\right) \pm Iy\right), \Re\left(\zeta\left(\Re\left(s'\right) \pm Iy\right)\right)\Re\left(\zeta\left(x' + \delta_x\right) \pm Iy\right), \Re\left(\zeta\left(\Re\left(s'\right) \pm Iy\right)\right)\right) < \varepsilon_Z\}$$

of my proposed complex open sphere and vice versa.

In other words, only the critical line s = 0.5 contains all non-trivial roots of Z = Re (ζ(s)) or Re (ζ(s)) = 0. On the other hand, all of Z = = εZ, which contains the meeting points of the real and imaginary parts of (s-δ) plus the line $Z = \varepsilon_Z$, must stay outside the critical line s = Re(s'). We have already visually and analytically proved that the Riemann Hypothesis is in practice correct.

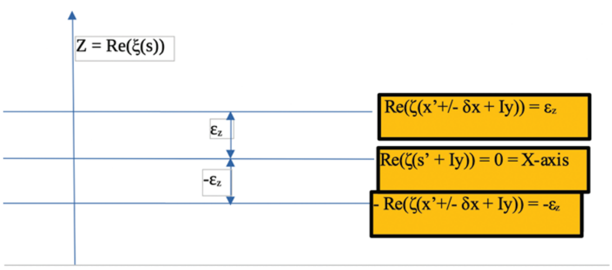


Figure 1: εx-δx complex open sphere concept for the proof to the fact that all non-trivial [6] zeta zeros must stay on the critical line [6] s = Re(s'); Re(ξ(s = Re(s') + Iy)) = 0 or the x-axis and the Re(ξ(s = x' +/- δx + Iy)) = when the ε → 0, Re(ξ(s = x' +/- δx + Iy)) - Re(ξ(s = Re(s') + Iy)) → 0. (N.B. The above proof of the εx-δx complex open sphere concept may be extended as the definition of a continuous function etc which will be discussed further in my series's next paper.)

In other words, given any $0 < \varepsilon_Z < 1$, we may select δ_x = such that Re(ξ(s = x' +/- δ_x + Iy)) - Re(ξ(s = Re(s') + Iy)) = $\varepsilon_Z \rightarrow 0$ but NOT EXACTLY equals to zero whenever ε_Z tends to zero. i.e. All other intersecting points for the real and imaginary parts plus the line of Z = ε_z, x' = Re(s +/- δ) must stay outside the x' = Re (s) = Re(s') (except those intersecting points plus the line Z =Re(ξ(s = Re(s')+ Iy)) = 0, at x' = Re (s) = s' which are just the roots of Z = Re(ξ(s = Re(s') + Iy)) = 0 or the critical line) and they are in fact the roots of Z = Re(ξ(s = Re(s') + I_y)) = ε_Z. Hence, the rest of the intersection points for the real and imaginary parts plus the line Z = Re(ξ(s = Re(s') + Iy)) = 0 at x' = Re (s) = Re(s') or the critical line must contain all of the non-trivial zeros. However, as shown in Figure 3 in the next page or the calculated result from the computer simulation, there is only one Z = 0 at Re(s) = 0.5 or Z = Re(ξ(s = 0.5 + I_y)) = 0 without any other alternative s = s' such that Z = Re(ξ(s = Re(s') + I_y)) = 0 where. This may induce a contradiction to the assumption that there are both and Re(s) = 0.5, which has the same properties as Z = Re(ξ(s)) = 0 (in general) together with what this writer has previously described about the ε-δ relationship between the Re(s') and the rest of the other Re(s''), which must NOT be Z = Re(ξ(Re(s) + I_y)) = 0. Hence, we have the confidence to conclude that Re(s') = 0.5 without any other choice of Re(s') staying in the Critical Strip Region 0 < Re(s') < 1. To proceed further, there must be one and only one critical line with Re(s) = 0.5.

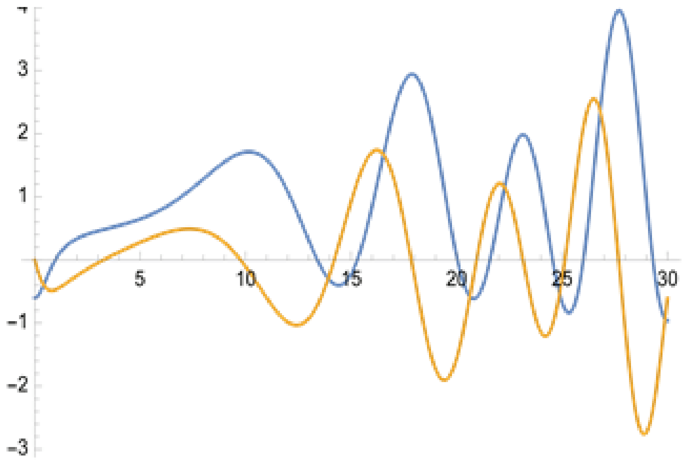


***Figure 2**: ξ(s) when s = 0.1 + I y*

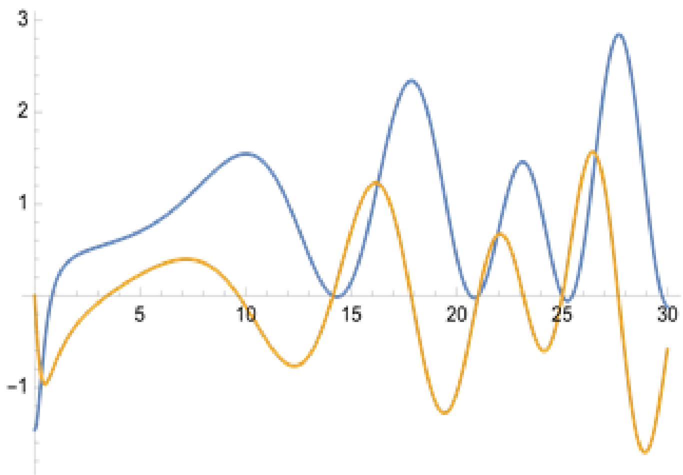


***Figure 3**: ξ(s) when s = 0.5 + I y*

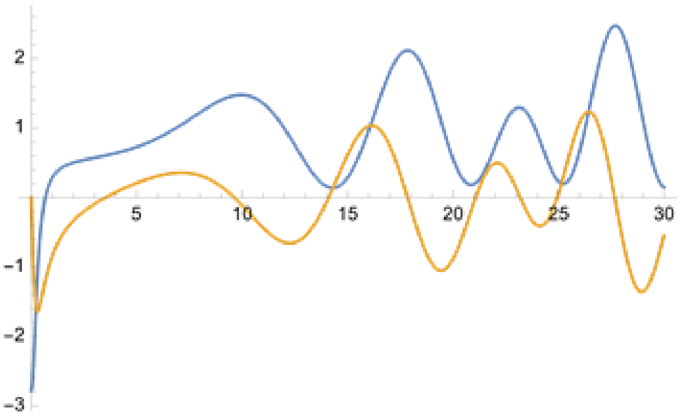


***Figure 4**: ξ(s) when s = 0.7 + I y*

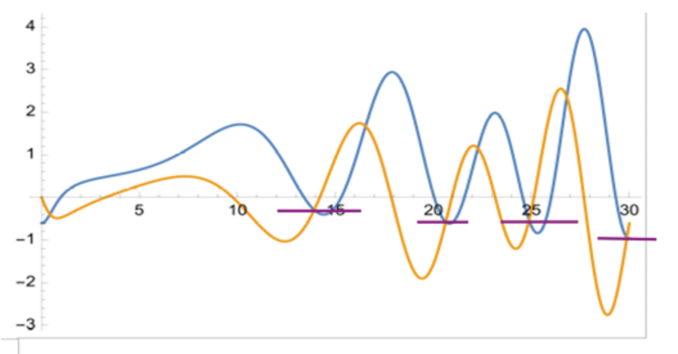
Note that for all other intersection points without meeting the line y = ε_Z or zero, we may still apply the above-described ε-δ concept and the graphical line (GL) intersecting method to find those points, but only with another value of ε'_Z. i.e. By moving the GL = ε upwards and downwards to meet the intersecting points of the real and imaginary parts of s in the zeta function: $\xi(s = x' + I_y + \delta) = \varepsilon_z$. Or, $\xi(s = x' + I_y + \delta) = \varepsilon_Z$ converges to $\xi(s = 0.5 + I_y) = 0$. Since for every root of $\xi(s = x' + I_y + \delta) = \varepsilon_Z$ whenever given any Z = ε_Z must converge to $\xi(s = 0.5 + I_y) = 0$, by the Squeezing Principle, the limit for all of the other roots in the $\xi(s = x' + I_y + \delta) = \varepsilon_Z$ is just the roots of $Z = \xi(s = 0.5 + I_y) = 0$. In fact, all the roots of $Z = \xi(s = 0.5 + I_y) = 0$ are the non-trivial zeta roots at Z = 0. Moreover, one may verify from Figure 3 directly or by the U.S.A. Mathematica programming software that all of the non-trivial zeta zero roots of $Z = \xi(s = 0.5 + I_y)$ are the roots of Z = 0. Hence, all the roots of $Z = \xi(s = 0.5 + I_y) = 0$ must be equal to all of the non-trivial roots of the zeta function. By the Sandwich Theorem, the convergent limit of all other roots for $\xi(s = x' + I_y + \delta) +/- \varepsilon_Z = 0$, or from

Figure 2: $\xi(0.1 + I_y)$ & Figure 4: $\xi(0.7 + I_y)$, must also tend to those non-trivial zeta zeros at s = 0.5 + I_y where x' = 0.5 and δ tends to a zero if we add a δ' and δ" to

s = 0.1 + I_y and s = 0.7 + Iy for every given ε''_Z and ε'''_Z respectively.

In brief, as this author has just proved:

- There is one and only one critical line x = 0.5 in the Critical Strip Region for all non-trivial Zeta zeros.
- The sandwiched convergent property from the upper and lower limits that all of the other real and imaginary intersection points must tend to the one and only one critical line x = 0.5;



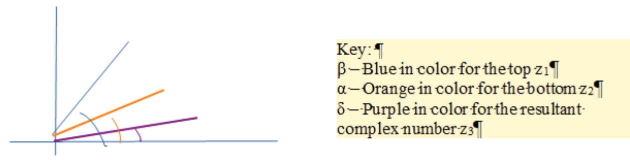
- On the other hand, a translation (x-axis) line may therefore be used to locate some other values of *x* (such as *x* = 0.1 or *x* = 0.7) such that the real parts of these complex numbers meet the imaginary parts. These complex numbers may thus form "another kinds of non-trivial zeta zeros" in the critical strip region with the same properties of those commonly known non-trivial zeta zeros lies on the critical lines *x* = 0.5. However, in this time, these "another kind of non-trivial zeta zeros may intersect with other values (e.g. *x* = 0.1 or *x* = 0.7) with their real and imaginary parts. Below is a graphical plotting for z = 0.1 + I*t:

Obviously, those purple lines are just the intersections of the real parts and imaginary parts, this author suggests that they may be another form "non-trivial zeta zeros" which are different from those of the normal non-trivial zeta zeros as:

Re(ζ (0.*x* + I*t)) = Im(ζ (0.*x* + I * t))   where *0.x* ≠ 0.5

In reality, the solved roots (i.e. those "abnormal non-trivial zeta zeros") of the above equation are just / only the subtraction between two angles of the complex number (0.*x* + I*t) relative to the original angles of the critical line complex zeta zeros (0.5 + I*t) or rotates the abnormal non-trivial zeta zeros clockwise relative to the real axis and also scales the abnormal one by the vice versa (or actually the reciprocal inverse) of the magnitude of the normal one. This is because:

For the complex number division, say (a + bi), we may need to multiply both of the top and the bottom by its conjugate (a-bi) and turns the denominator into a real number. Thus, the result is a multiplication at the top which is certainly a rotation in a clockwise direction



Hence, the intersection/meeting points of that represents these "abnormal non-trivial zeta zeros" in some sense are just / only the virtual / false roots for the Riemann Zeta function (i.e. the non-trivial zeta zeros lie on the critical line X = 0.5) as what have found in the previous section with the (same and consistent) result. Actually, by collecting as much as possible of these rotating angles and using the method of interpolation, we may further forecast the next or even more coming intersecting points etc. In fact, we may factorize the sine function and may finally obtain an impulse function etc. By reverse engineering, we may thus compute back the corresponding Riemann Zeta Non-Trivial Zeros. In practice, there may be a symmetric group between these "abnormal non-trivial zeta zeros" and normal non-trivial zeta zeros which may be further studied within an area of cryptography (encryption and decryption algorithm etc).

i.e. $\frac{e^{i2m\pi}}{e^{ik\pi \pm in\theta}}$ = 0 - x for n = 0,1,2…,n-1

or

$$ln\,(1) + ln\,(-x) = ln(1) + ln\,(-1) + ln\,(x) = i(2m-k)\pi \pm in\theta$$

But $ln(-1) = i\pi$ and $ln(1) = i2*\pi$-- an algebraic modulo group etc. That is in general, $ln(x) = (2m-k-1)\pi i + n\theta i$ where k = 1,2…,*n* & *n* = 0,1,2,...(*n-1*). Or $ln(x) = [2m-(k+1)]\pi i + n\theta i$,

i.e. $ln\,(x)\ mod\ \pi i. \equiv .n\,\theta\, i$. Or

1 mod $\frac{\pi i}{ln(x)} . \equiv . \frac{n\theta i}{ln(x)} . = .e^{\frac{\pi i}{ln(x)}-1} . = .e^{\frac{n\theta i}{ln(x)}-1}$ provided that $\frac{n\theta}{ln(x)}$ is a prime number (by the Fermat's Little Theorem) or guess the nearest prime number from the prime counting function (as $\frac{n\theta}{ln(x)}$ may be used to approximate $\pi(x) = \frac{x}{lnx}$ which can be used to generate the public key(s) and the private key(s) of the RSA Encryption/Decryption etc. In fact, $\frac{x}{lnx} = \pi(x)$ which is just the prime counting function or the Prime Number Theorem

Also, $\frac{x}{lnx} = \frac{x}{(k+1)\pi i + n\theta i}$ and when $(k+1)\pi i + n\theta i \to 0, \frac{x}{(k+1)\pi i + n\theta i} \to \infty$ which is just the impulsive encryption or the chaos system for dynamic key generation.

According to the Riemann Explicit Formula, we may have:

For any given non-trivial Zeta Zeros of complex number, say $0.5 + y^*i$, $0.5 + y^*i = \sum$ *prime countingfunction*, thus we may conclude that:

encrypt/decrypt ↔ prime numbers guess from $\pi(x)$ ↔ $\frac{x}{(k-1)\pi i + n\theta i}$ ↔ approximate prime counting function ↔ select any two guessed prime numbers, say p and q to generate the public key n = p*q ↔ anyone may encrypt a message by n without knowing p & q ↔ decrypt a message by the sender who knows the primes p & q (or the impulsive encryption i.e. the chaos dynamic encrypt/decrypt).

All in all, these so-called "abnormal non-trivial zeta zeros" are just the rotation of angles to the normal non-trivial zeta zeros. In other words, we, authors, Lam and Siu have shown that all of the "abnormal non-trivial zeta zeros" are just the normal non-trivial zeta zeros, they are actually the same. Therefore, we, Lam and Siu have shown that there is a contradiction to the assumption there was another non-trivial zeta zeros other than the (0.5 + I*t) lying on the critical line. In reality, both of the positive rotational angle and the negative rotational angle to the normal non-trivial zeta zeros constitute a sandwich to the critical line x = 0.5. Or a shift from both of the left and right hand-side for approaching the critical line must lead to the fact that those of the "abnormal non-trivial Zeta zeros are actually the same as the normal non-trivial zeta zeros or the vice versa (the mirror image inverse). Hence, we both author, Lam and Siu conclude that the Riemann Hypothesis is in fact correct or all of the non-trivial zeta zeros are just lying on the critical line x = 0.5 and all other alternatives are only the rotational angles of these "normal non-trivial zeta zeros".

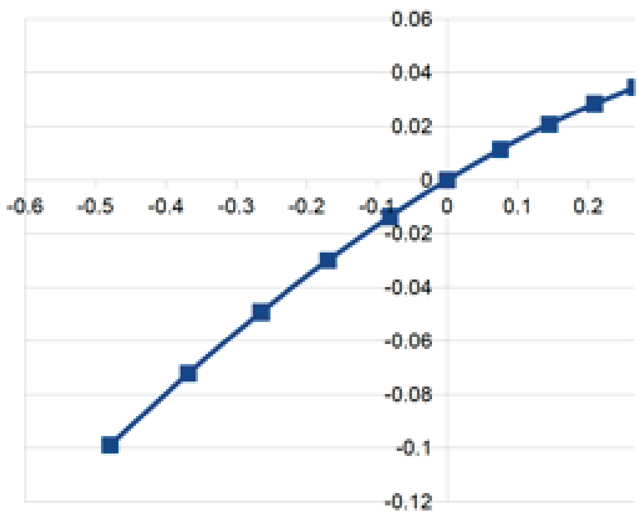
As the present research is just a private one, this author (whose role is just to heat the fire head for the study) has only limited resources for it and will leave to those interested national parties such as the University of Hong Kong's Institute of Mathematics Research or the Liu Bie Ju Centre for Mathematical Science of the City University of Hong Kong or some other countries' Mathematical research centre / institute like the Institut des Hautes Ètudes Scientifiques in France and the American Mathematics Society & Mathematical Society of Japan etc.

(N.B. In order to make the above new encryption/decryption theory more than perfect, it is wise for us to employ those nearest prime numbers as the input data to construct an Artificial Intelligent model such that the data encrypted/decrypted will be in an optimized status and hence improve the efficiency. However, the A.I. modeling research is out of the focus in the present paper, those interested parties may be required to reference some of my previous or former series of papers in the topic of HKLam Theory such as the case in the climate and virus infection etc. The optimization contents are in fact contained in another papers of the present Riemann Hypothesis series. Therefore, this author will NOT repeat another chained cycles in the same subject or topic.)

(N.B. 0 < x' < 1 which stays in the critical strip region or s are, in fact, those non-trivial zeros [5] for the Riemann Zeta function [5] if $\xi(s) = 0$ where $s = x' + I_y + \delta$. Obviously, what the trivial zeros for the Riemann Zeta function is $\xi(-2) = \xi(-4) = \ldots = 0$ which are outside the critical strip 0 < Re(s) <1.)

Conclusion

To conclude, in this paper, the writer has successfully found a duality for the Birch & Swinnerton-Dyer Conjecture which is useful for the description of the set of rational solutions to equations defining an elliptic curve. Furthermore, assume on the contrary, there is another critical line rather than x = 0.5 which should contain all of the non-trivial Zeta Zeros. With the help of the computer simulated program segment and its associated graphics, these figures are sandwiched from the right and left hand limit to a zero together at x = 0.5. Also with a comparison of graphics at the pole of x = 1 with a movement of just one direction which may imply that there is only one critical line at x = 0.5 which must constitute a contradiction to the assumption with two critical lines. Finally, there is also another proof to the Riemann Hypothesis which shows there must be one and only one critical line at x = 0.5 together with the sandwiched convergent ε-δ property from the upper and lower limits that all of the real and imaginary intersection points must tend to the one and only one critical line at x = 0.5. From the graphs Figures 1-4, in the geometry sense [9], other values in the critical region {x | 0⩽x⩽1, x≠0.5} after the transformation: Re(Zeta(x+yI)) & Im(Zeta(x+yI)) beside the critical strip (x=0.5) of the Riemann hypothesis is just a shift of a vertical and horizontal δ and ε amount. In addition, the non-trivial zeros of Re(Zeta(0.5 + I*t)) = Im(Zeta(0.5 + I*t)) when t = nth non-trivial zeta zeros takes the same role as those other values' (where x≠0.5) intersection points with x = 0.5 (is the most optimal one or minimum/ maximum after an integration will give a concave or convex shaped parabola upon to their respective slopes) as Re(Zeta(0.5 + I*t)) = Im(Zeta(0.5 + I*t)) = 0 when t = nth non-trivial zeros. But the other points (where x ≠ 0.5 and { 0 ⩽ x ⩽1}), R(Zeta(x + I*t)) = Im(Zeta(x + I*t)) = (ε, δ) or just a shift. Obviously, other points do NOT achieve the property equals to a zero as x = 0.5. Certainly, the mirror image inverse of the above scholar outcome is also true. In other words, by equating the Re(Zeta(0.5 + I*t)) = Im(Zeta(0.5 + I*t)) = 0, we may solve back those non-



trivial zeta zeros. Or by fixed a value of x', (where x'≠0.5 and { 0 ⩽ x' ⩽ 1}), and Re(Zeta(x' + I*t)) = Im(Zeta(x' + I*t)), we may get back the solved intersection points' values. In reality, the aforementioned Maple Soft computation will be left to my last conclusive section in the present research paper series. Thus, we can conclude that both of the contradictory proofs are used to show that there is one and only one critical line x = 0.5 from left to right and the upper to lower. The results imply that the Riemann Hypothesis can be true. Alternatively, RH may be false in the sense that other "abnormal non-trivial zeta zeros" is just the angular rotation of those normal non-trivial zeta zeros.

```
w := InverseLaplaceTransform[1/(4 + s), s, 0.3]
helper[c0_, c_] :=
 Module[{r = (Abs[#] < 2 & /@ c)}, Pick[#, r] & /@ {c0, c^2 + c0}]
z[n_Integer?Positive, c0_List] := z[n - 1, c0, c0^2 + c0 + w]
z[0, c0_, _] := c0
z[n_, c0_, c_] := z[n - 1, Sequence @@ helper[c0, c]]
With[{stepSize = .0001, n = 46},
 Module[{c0, p},
  c0 = Flatten[
    Table[x + I*y, {x, -1.42, -1.39, stepSize}, {y, -.005, .025, step-
Size}]];
  p = z[n, c0];
  ListPlot[Transpose[{Re[p], Im[p]}]]]]
```

In fact, for $y = kx$ [10]; $\int kxdx = \frac{kx^2}{2} + c$ (may be a fractional mandelbrot set with impulse function [11]). In practice, the above function is used for the both of the non-linear system analysis and the chaos theory etc) which is in fact a parabola passing through the zero (as the optimum or minimum at x = 0.5, I.e. concave in shape) with the y-intercept constant value c. Certainly, there is also its conjugate with the similar convex shaped maximum etc. Actually, in the sense of French philosophy education or the vice versa way, there is an impulse function that can be incorporated into the Mandelbrot set generation process for the creation of the so-called "fractional Mandelbrot sets with impulse". The delta function is defined by:

$$z_{n+1} = z_n^{\ 2} + c + impulse(n - N)$$

where N is the number of iteration n is the n-th term of the complex variable z.

In terms of (a simple) Mathematica programming code, we may have:

To go ahead a step, we may then enhance a deeper algorithm like the following:

1. Set up a function for the inverse Laplace Transform of a step function such as ;
2. Establish a recursive function for the Mandelbrot set;
3. Numerically inverse Laplace Transform the recursive function in Step 2;
4. Add the above numerically inverse Laplace Transform function with the inverse Laplace Transform of the step function in the Step 1;
5. Table the values in Step 4 and interpolate a function together with the Laplace Transform to obtain the plotted "Impulse driven Mandelbrot set" of figure.

Last but not least, we may still evaluate (regularize) the partial sum of the Riemann Zeta function as defined by (for N=50, x=25):

$$2\sum_{k=1}^{N}\frac{1}{k} = \left\{\frac{1}{N-1} + 1\right\} + \left\{\frac{1}{N-2} + \frac{1}{2}\right\} + \left\{\frac{1}{N-3} + \frac{1}{3}\right\}$$

$$+ \ldots + \left\{\frac{1}{N-(N-1)} + \frac{1}{N-1}\right\} + 2\frac{1}{N}$$

$$= 2 + \sum_{k=1}^{N}\frac{-1}{k(x-k)}$$

$$= 2\frac{1}{N} + \oint\frac{-1}{k(x-k)}dk$$

$$= 2\frac{1}{k} - Res((C_x^k), k = 50, x = 25)$$

.≈. $1.209656216*10^{13}$

For N = 49, x = 24,

$$2\sum_{=1}\frac{1}{k} = \frac{1}{49} - Res((C\ ), k = 49, x = 24)$$

.≈. $3.1602652*10^{13}$

Hence, $\sum_{k=1}^{N}\frac{1}{k} = 6.02482811*10^{12}$ or $\sum_{k=1}^{N}\frac{1}{k} = 3.1602652*10^{13}$ depends on the values of the variables k and x which are two different results for the odd function and even function parts of the Riemann Zeta function and is actually the harmonic progression. Actually, in terms of area, we may get:

$$\sum_{k=1}^{N}\frac{1}{k^2} \leqslant \sum_{k=1}^{N}\frac{1}{k} \leqslant \sum_{k=1}^{N}\frac{1}{lnk}$$

For N→∞, $\sum_{k=1}^{N}\frac{1}{k^2} \to \frac{\pi^2}{6}$ and $\sum_{k=1}^{N}\frac{1}{lnk} \to \pi(x)$ which is just the prime counting function.

i.e. $\frac{\pi^2}{6} \leqslant \sum_{k=1}^{\infty}\frac{1}{k} \leqslant \pi(x)$

(N.B. $\frac{\sum_{k=1}^{N}\frac{1}{k}}{N} = \sum_{k=1}^{N}\frac{1}{k} = \sum_{k=1}^{N} lne^{\frac{1}{x_k}}\Delta x = \int_0^1 lne^x dx = \int_0^1 xdx = \frac{1}{2}$

i.e. $\sum_{k=1}^{N}\frac{1}{k} = \frac{1}{2}N$

Or the prime counting function is just the upper bound of the Riemann Zeta function. To be precise, we may use the prime counting function as an approximation to calculate the area under the curve of the Riemann Zeta Function. In the same manner, we may also apply the revolting prime counting function to approximate the revolting surface area of the Riemann Zeta function or the logarithmic function. Thus, in a deductive way, we may also establish a respective toy black hole model with the layered prime numbers as the quantization corresponding to my previously proposed toy black hole model that is quantized by the layered of the non-trivial zeta zeros etc. As the prime numbers may be viewed as just the Fourier transform of the non-trivial zeta zeros, therefore, one may consider such kind of the prime number toy model black hole as either the time or frequency signal proportion part of the non-trivial zeta zeros toy model black hole. Obviously, the prime number toy model black hole or in general known as the "Parallel Universe" may NOT be the conjugate part of the non-trivial zeta zeros toy model black hole or the so called "white hole". In reality, from the fact of the parallel universe, we may guess back the actual situation of the present universe etc. Theoretically, the conjugated white hole should be formed by the quantized layered complex conjugates of all those non-trivial zeta zeros. In fact, both of the non-trivial zeta zeros toy black hole and its complex conjugate toy white hole may be connected with the toy model worm hole. At the same time, the Fourier transform of the original real observed black-hole toy model will give the prime-numbered black-hole toy model and its conjugated white hole toy model where they may be linked with the worm-hole. Some of the astrophysicists believe that these quantized layered wormholes may be the key for a spacecraft to travel from one universe to another universe in a multi-universes model (or the multiverse). Similarly, in some schools of quantum physics, people may believe that the relationship between momentum and the position of a particle is just the Fourier transform of each other. Hence, once we may have observed either the values of momentum or position for a particle, we may compute the other conjugate paired one. The implications may be the microscopic black hole, the Compton-Schwarz-child Correspondence, the sub-Planck Black hole for an exploration of the nature of the quantum gravity and the connection among the microscope and macroscopic world etc. In a simplified and a small scale manner as an analog to the complicated singularity of the real black hole, let us consider the inverse Laplace transform of the singularity function that is in the form of the impulse function $\frac{1}{s}$, the $\mathcal{L}^{-1}\left(\frac{1}{s}\right)=1$.

In practice, $\mathcal{L}^{-1}\left(\frac{1}{s}\right)=1$. is just the standard unit of time in the time domain and is significant for the linking between the quantum mechanics and the general relativity. Then we may compare with the previously found quantum-relativity standard Planck's time [] for an adjustment so as to get a more accurate universe standard quantum-relativity unit time. Such kind of research in the calibration of the standard universe time may be left to the next stage of my investigation if there may be any. Certainly, if there may be really a standard universe time, we may then in the sense of the "reverse engineering" to further discover more detailed and complete truths about the presently so-called "black-hole singularity" or even the theory for quantum gravity. Lastly, if we consider the Riemann Hypothesis is true (or by our human design), then there may be an infinite number of Riemann Spheres (from both of the North & South poles) lying on the critical line with the stereographic projection on a straight line of the associated complex plane or in the vice versa. Details will be described in my final & conclusive paper of the present Riemann non-trivial Zeta zeros series []. At the same time, if we further differentiate the constant "1" from the inverse Laplace transform of (that this author has shown before), we may get the "0". This is just the origin in the complex plane coordinate of the stereographic projection in the previously discussion. In general, there may be lots of "impulse" all around the solving solutions of the zeta function which can be used as control system for the commercial digital signal processing in the field of engineering etc. Actually, these impulse(s) may be turned into the prime numbers through the Fourier transform like method such as the Riemann Explicit Formula etc or in the vice versa (the mirror inverse) way.

In a nutshell, we may establish one non-trivial zeta zeros black hole toy model and one prime number black hole toy model from the observed data of a real black hole. Actually, these non-trivial zeta zeros and the prime number black holes are in fact the Fourier transformed pair of each other respectively which may be corresponding to the time and frequency domains of the investigated targeting real black hole. The analytical results may then help us further develop the mystery theory in quantum gravity together with a forward step for the theory of everything.

To sum up, we may have:

- A Point is lying on the line between 0 and 1 excluding 0.5 (says 0.1 + t1*I or 0.7 + t2*I);
- A point (or the non-trivial zeta zeros), 0.5 + t3*I is lying on the critical line;
- Consider the two vertical lines (say either x = 0.1 or x = 0.7) and x = 0.5;
- These two lines contain any two pints lying on the complex plane are just the angle rotation with each other;
- Hence, by the Sandwich Theorem, when the both positive and negative (with the same angular rotations) of the abnormal non-trivial zeta zeros approaching each other which are lying on the two upper and lower lines width of radius (-ε, ε), they will tend to the limit and meet at the middle or just at the zero angle. In fact, the zero is just the critical line x = 0.5 or the normal non-trivial zeta zeros. The vice versa (or the mirror inverse) for the positive and negative rotational angles on the corresponding upper and lower lines with width of radius (-ε, ε) between non-trivial zeta zeros on the critical line and the abnormal non-trivial zeta zeros is also true. .

i.e. Given any

$(\varepsilon_z, \varepsilon_y) = d\,(\mathfrak{R}\,(\zeta\,(x' \pm \delta_x \pm Iy \pm \delta_y\,)), \mathfrak{R}\,(\zeta\,(\mathfrak{R}\,(s') \pm \delta_x \pm Iy))),$

there will always be a (δ_x, δ_y) with

$(\delta_x, \delta_y) = (\mathfrak{R}\,(\zeta^{-1}\,(\pm\varepsilon_Z + \mathfrak{R}\,(\zeta\,(x' - \mathfrak{R}\,(s') \pm Iy))),(\pm\varepsilon_y + (Iy - Iy))))$

where $(\delta_x, \delta_y) \rightarrow (|\,(x' - \mathfrak{R}\,(s'))|, |(y - y)|)$ when $(\varepsilon_Z, \varepsilon_y) \rightarrow (0,0)$.

and also $\frac{e^{i2m\pi}}{e^{ik\pi \pm in\theta}}$ - x for n = 0,1,2…,n-1 lies on the upper line $\mathfrak{Re}(\zeta\,(x' \pm \delta_x + Iy)) = \varepsilon_z$ and the lower line $\mathfrak{Re}(\zeta\,(x' \pm \delta_x + Iy)) = -\varepsilon_z$ with width $(-\varepsilon_z, \varepsilon_z) \rightarrow (0,0)$ and sandwiched to approaching the middle critical line x = 0.5. Or

$0 = -\varepsilon_z = -\,\mathfrak{Re}(\zeta\,(x' \pm \delta_x + Iy))\ .\le.\ \mathfrak{Re}(\zeta\,(\mathfrak{Re}(s') + Iy))\ .\le.\ \mathfrak{Re}(\zeta\,(x' \pm \delta_x + Iy)) = \varepsilon_z = 0$

Then $\mathfrak{Re}(\zeta\,(\mathfrak{Re}\,(s') + Iy)) = 0$ (X-axis) or $\mathfrak{Re}(s') = 0.5$, i.e. the critical line x = 0.5.

(N.B. If a function $|\ f(x) - f(y)| \ \le |\ (x - y)^2|$, then f(x) is a constant. In fact, the integral of a constant function is a linear function or a linear regression, i.e. a statistical correlation may exist. Also, if we rearrange the above equation, there may also be an impulse in the field of the commercial engineering etc.)

The aforementioned steps show that all of the intersection points (of the Re and Im) for the other lines (says 0.1 or 0.7) are just the angular rotation (with positive or negative rotating) to the intersection points (Re and Im) of the critical line x = 0.5. At the same time, the normal non-trivial zeta zeros (intersection points of the Re and Im) at the critical line x = 0.5 can be rotated positively and negatively to achieve the abnormal non-trivial zeta zeros (the intersection points of Re and Im) for the other lines (says either x = 0.1 or x = 0.7). We may conclude that abnormal non-trivial zeta zeros are just the normal non-trivial zeta zeros or the vice versa (or the mirror image inverse) and this completes the summary to the proof of the Riemann Hypothesis.

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