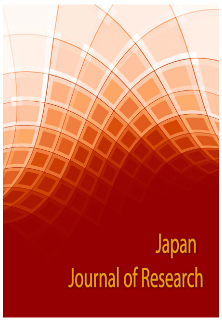
Japan Journal of Research

Original Article



Dispersion Relation for 2D Transverse Magnetic (TM) Plasmonic Wave: Analytic Versus FEM

Arslan Asim[1], Michael Cada[1,2] and Yuan Ma[1]

[1]Department of Electrical and Computer Engineering, Dalhousie University, Halifax, Canada.

[2]Faculty of Materials Science and Technology, VSB-Technical University of Ostrava, 17. listopadu 2172/15,708 00 Ostrava-Poruba, Czech Republic.

Correspondence

Arslan Asim

Department of Electrical and Computer Engineering, Dalhousie University, Halifax, Canada.

Abstract

A metal-dielectric interface has been studied in this paper in the light of wave propagation characteristics. A new analytical result, through a simpler approach for deriving the dispersion equation in plasmonic waveguides, has been reported and visualized. A two-dimensional Transverse Magnetic (TM) wave is theorized and analyzed with respect to its propagation along the boundary of metal and air. Mathematical analysis results in the relationship between the propagation parameter and wavelength of operation. The plots reveal good conformity with the previous studies. For comparison, the propagation characteristics in free space have also been included. To validate the proposed results of this study, Finite Element Analysis (FEA) has been used to visualize the distribution of fields near the metal-dielectric boundary.

- Received Date: 31 Oct 2024
- Accepted Date: 10 Nov 2024
- Publication Date: 16 Nov 2024

Introduction

In this paper, an exact solution to the dispersion equation of a plasmonic wave has been presented. The final form of the achieved expression is different from the previously reported forms [1,2]. The mathematical analysis assumes a two-dimensional Transverse Magnetic (TM) plasmonic wave propagating along the boundary of a metal and air as depicted by Figure 1. A detailed analytical procedure, starting from Maxwell's equations, accompanies the final result. The result has also been plotted for different metals to validate its accuracy. The novelty of the paper lies in the analysis of an existing problem with a more rigorous and elaborate mathematical approach followed by a new analytical result.

Figure 1. *Plasmonic wave propagating along the boundary of two materials.*

Equations for two-dimensional time harmonic TM waves propagating along the interface are shown below. The electric and magnetic field vectors $\vec{E}$ and $\vec{H}$ show magnitudes $\left(E^{x}_{M,A}, E^{z}_{M,A}, E^{y}_{M,A}\right)$ as well as directions *(x ̂, ŷ, ẑ)* of the fields. In the expressions below, the subscripts M and A stand for points in the metal and air respectively while superscripts are the dimensions along which the magnitudes are oriented.

$$\vec{E} = E^{x}_{M,A} e^{-i\omega t} e^{i\gamma_{M,A} x} e^{i\beta z} \hat{x} + E^{z}_{M,A} e^{-i\omega t} e^{i\gamma_{M,A} x} e^{i\beta z} \hat{z} \quad (1)$$

$$\vec{H} = H^{y}_{M,A} e^{-i\omega t} e^{i\gamma_{M,A} x} e^{i\beta z} \hat{y} \quad (2)$$

Using Maxwell's equations, we arrive at equations 5 and 6.

$$\vec{\nabla} \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t} \quad (3)$$

$$\frac{\partial H^{y}}{\partial x} - \frac{\partial H^{x}}{\partial y} = i\gamma H^{y} = -i\omega\varepsilon E^{z} \quad (4)$$

$$\beta H^{y} = \omega\varepsilon E^{x} \quad (5)$$

$$\gamma H^{y} = -\omega\varepsilon E^{z} \quad (6)$$

In the steps above, the time derivative *∂/∂t* yields the term -iω while spatial derivatives *∂/∂x* and *∂/∂z* yield *iγ* and *iβ*, respectively. Introducing boundary conditions provides the three relationships in (7). Parallel field components must be continuous across the boundary. The normal component of the electric displacement $D = \varepsilon E$ is continuous at the boundary ($x = 0$).

$$E^{Z}_{M} = E^{Z}_{A}, \quad H^{y}_{M} = H^{y}_{A}, \quad \varepsilon_{M} E^{x}_{M} = \varepsilon_{A} E^{x}_{A} \quad (7)$$

For x > 0 (x = 0 lies along the interface),

$$\vec{H}_{A} = H^{y}_{A} e^{i\beta z} e^{i\gamma_{A} x} e^{-i\omega t} \hat{y} \quad (8)$$

and, for

$$\vec{H}_{M} = H^{y}_{M} e^{i\beta z} e^{-i\gamma_{M} x} e^{-i\omega t} \hat{y} \quad (9)$$

Equation (6) can be realized for both metal and air to achieve equations (10) and (11).

$$\gamma_{M} H^{y}_{M} = -\omega\varepsilon_{M} E^{Z}_{M} \quad (10)$$

Citation: Asim A, Cada M, Ma Y, Dispersion Relation for 2D Transverse Magnetic (TM) Plasmonic Wave: Analytic Versus FEM. Japan J Res.2025;6(1):086

$$\gamma_A H_A^y = -\omega\varepsilon_A E_M^Z \qquad (11)$$

Using boundary conditions and dividing equation (10) by (11)

$$\frac{\gamma_M}{\gamma_A} = \frac{\varepsilon_M}{\varepsilon_A} \qquad (12)$$

Now, the curl of the Maxwell's equation in (3) and (4) gives the following results.

$$\vec{\nabla}\times\vec{\nabla}\times\vec{H} = \vec{\nabla}\times(-\varepsilon i\omega\vec{E}) \qquad (13)$$

$$-\nabla^2 H = -\varepsilon i\omega\vec{\nabla}\times\vec{E} \qquad (14)$$

$$-\left[(-i\beta)^2\vec{H} + (-i\gamma)^2\vec{H}\right] = -\varepsilon i\omega\vec{\nabla}\times\vec{E} \qquad (15)$$

$$\left(\beta^2+\gamma^2\right)\vec{H} = +\omega^2\varepsilon\mu_0\vec{H} \qquad (16)$$

$$\beta^2+\gamma^2 = +\omega^2\varepsilon\mu_0 \qquad (17)$$

Equation (17) can be applied separately to the two media under consideration i.e. metal and air.

$$\gamma_M^2 = -\beta^2 + \omega^2\mu_0\varepsilon_M \qquad (18)$$

$$\gamma_A^2 = -\beta^2 + \omega^2\mu_0\varepsilon_A \qquad (19)$$

The expressions in (18) and (19) can be used to replace the left-hand side expressions in (12).

$$\frac{-\beta^2 + \omega^2\mu_0\varepsilon_M}{-\beta^2 + \omega^2\mu_0\varepsilon_A} = \frac{\varepsilon_M^2}{\varepsilon_A^2} \qquad (20)$$

By rearranging the equation (20), β^2 can be isolated.

$$(-\beta^2 + \omega^2\mu_0\varepsilon_M)\varepsilon_A^2 = (-\beta^2 + \omega^2\mu_0\varepsilon_A)\varepsilon_M^2 \qquad (21)$$

$$\beta^2 = \omega^2\mu_0\left(\frac{-\varepsilon_A^2\varepsilon_M + \varepsilon_M^2\varepsilon_A}{\varepsilon_M^2 - \varepsilon_A^2}\right) \qquad (22)$$

Since $\varepsilon_M = \varepsilon_0\left(1-\frac{\omega_p^2}{\omega^2}\right)$ and $\mu_0\varepsilon_0 = \frac{1}{c^2}$,

$$\beta^2 = \frac{\omega^2\mu_0\left\{-\varepsilon_0^2\varepsilon_0\left(1-\frac{\omega_p^2}{\omega^2}\right) + \varepsilon_0^2\left(1-\frac{\omega_p^2}{\omega^2}\right)^2\varepsilon_0\right\}}{\varepsilon_0^2\left(1-\frac{\omega_p^2}{\omega^2}\right)^2 - \varepsilon_0^2} \qquad (23)$$

Rearranging equation (23) leads us to (25). Quadratic formula can then be used to find solutions for ω^2.

Table 1. *List of symbols used in the paper.*

NOTATION	DESCRIPTION
$\vec{E};\vec{H}$	Electric, Magnetic Field Vectors
$E_{M,A}^y;H_{M,A}^y$	Electric, Magnetic Field Components in Metal (M)/ Air (A) along y (or x or z) axis
ε_0	Permittivity of free space
μ_0	Permeability of free space
c	Speed of light in vacuum
β	Propagation constant along z-direction
γ	Propagation constant along x-direction
ω_p	Plasma frequency
N	Electron concentration
q	Electron charge
m_0	Electron mass

$$\beta^2c^2(\omega_p^2 - 2\omega^2) = \omega^2\omega_p^2 - \omega^4 \qquad (24)$$

$$\omega_{1,2}^2 = \frac{\omega_p^2}{2} + \beta^2c^2 \pm \sqrt{\frac{\omega_p^4}{2} + \beta^4c^4} \qquad (25)$$

The new result in equation (26), reveals that the propagation of a two-dimensional plasmonic wave depends on the propagation constant along z-direction, plasma frequency and speed of light in vacuum. It is independent of the other propagation constant, γ.

The result in (29) has been plotted for silver (Ag), copper (Cu) and aluminum (Al) as shown in Figure 2. The values for electron densities [3] of the three metals are given in Table 2, where $\omega_p = q\sqrt{\frac{N}{m_0\varepsilon_0}}$

Table 2. *List of symbols used in the paper.*

Metal	N (1/m³)
Ag	5.86×10^{28}
Cu	8.47×10^{28}
Al	18.1×10^{28}

Next, a plasmonic waveguide has been modeled and simulated in two dimensions through COMSOL Multiphysics, which employs Finite Element Method (FEM). In the derived equations above, air has been used as dielectric. Therefore, air has been chosen as the dielectric for the simulations too. The physical mechanisms of the study relate to 'Electromagnetic Waves, Frequency Domain (emw)'. The structure is investigated by means of boundary mode and frequency domain analyses. The top and bottom layers of the waveguide are assigned scattering boundary conditions. A TM wave is injected into the waveguide at port 1. It exits at port 2 after generating plasmon resonance. Figure 3 shows the electric field distribution around the metal-dielectric interface at a wavelength of approximately1500nm, which signifies telecom band. The fields are confined in that region giving rise to surface plasmons. The mesh size for numerical analysis is selected by the physics governing the study. However, a user-defined mesh is introduced near the metal-dielectric boundary. The user-defined mesh is much finer than the physics-controlled mesh. Single element size for each mesh varies by a factor of 100. To model Ag in COMSOL, Drude model for electric wave propagation has been used. The model uses the following values: plasma frequency = 14.0×10^{15} s^{-1} and damping constant = 0.032×10^{15} s^{-1}. A parametric sweep is used to observe the effects of electromagnetic propagation with varying frequency.

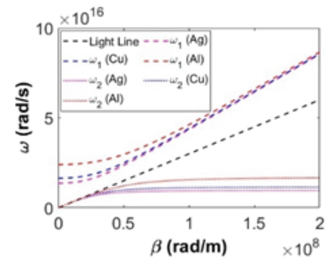


Figure 2. *Plots showing the relationship between propagation constant (β) and frequency (ω).*

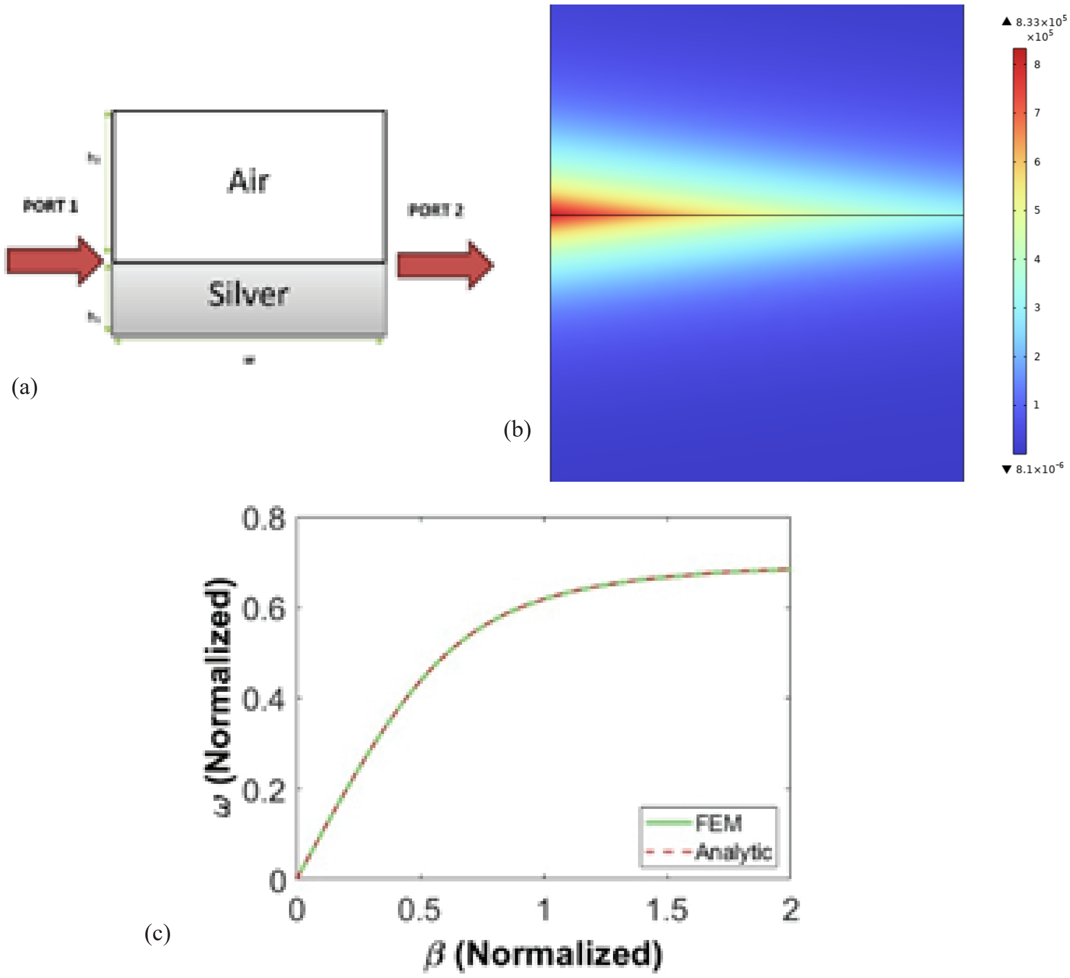


Figure 3. *(a) Finite Element Analysis (FEA) setup. (b) Normalized electric field distribution (V/m) at the silver-air (metal-dielectric) interface showing the characteristic Surface Plasmon Polariton (SPP) wave. (c) Dispersion curves for analytical and simulation methods. The normalized parameters represent* $\beta c/\omega_p$ *and* ω/ω_p *on the x and y axes respectively.*

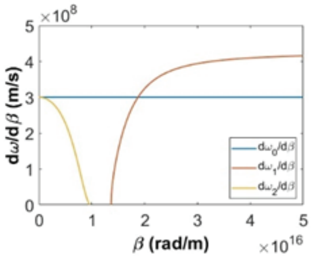


Figure 4. *Graphs showing the relationship between propagation speed (v = dω/dβ) and propagation constant (β).*

The FEM analysis provides only one curve for the dispersion relation. The curve generated by COMSOL is compared with the one generated through the exact solution (by MATLAB). The two curves fit almost perfectly. The comparison has been carried out using normalized parameters ($\beta c/\omega_p$ and ω/ω_p) for Ag.

However, it is imperative to understand why FEM analysis does not provide both the solutions that we have acquired from the analytic method. So, the dispersion curves in Figure 2 have been differentiated using 'gradient' command in MATLAB and plotted in Figure 4. The relations for which derivates have been plotted include the light line (free space/ air), ω_1 and ω_2 for Ag. The derivative $d\omega/d\beta$ yields the speed of the wave. For ω_1, the wave velocity exceeds the speed of light in free space after a specific value of β, which is unrealistic. Hence, one of the two analytic solutions must be rejected for the physical realization of plasmonic waveguides.

This paper approaches the classical surface plasmon theory from a mathematical perspective and aims to present a novel result for the dispersion relation of a single metal-dielectric layer. The new result is supported by a rigorous analytical methodology utilizing Maxwell's equations and fundamental boundary conditions. The result has been visualized to validate its accuracy.

Acknowledgements

This work was supported by the Natural Sciences and Engineering Research Council (NSERC) of Canada. The work of Arslan Asim was supported by Research Nova Scotia (RNS) and Dalhousie University.

References

1. Cheng JC, Liaw JW, Ma C, Zhang R. Dispersion relation of a plasmonic waveguide on substrate. Vacuum. 2015; 118:26–31. doi: 10.1016/j.vacuum.2014.08.018.
2. Khattak A. Plasmonic Slab Waveguides: Theory & Application for Sensors," p. 2020, 2020, [Online]. Available: https://scholars.wlu.ca/etd/2280.
3. Ashcroft N, Mermin N. Solid State Physics. Orlando, FL: Saunders College, 1976.