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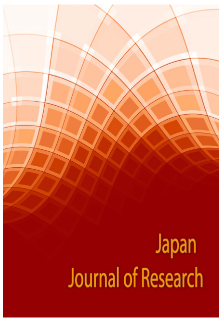

Analytic Solution of Steady Heat Conduction with Transverse Convection and Radiation Heat Transfer

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Abstract

An exact analytic solution to the problem of steady heat conduction with convective and radiative heat transfer normal to the conduction heat flow is presented. The solution is unique as it does not impose any assumptions on the surroundings and fluid temperature values and addresses all possible tip boundary conditions. The temperature profiles in the direction of heat conduction are produced for constant temperature boundary condition at the base and three different boundary conditions at the tip: adiabatic, constant temperature and radiative/convective heat transfer. Approximate solutions to the implicit exact solution are also developed. The analytic solutions, exact and approximate for the adiabatic tip boundary condition, compare very well to experimental data.

Introduction

The problem of steady state, one-dimensional heat conduction with transverse heat transfer by convection and radiation has many different applications; the most prominent perhaps being the use of extended surfaces for cooling, i.e., fins. It produces a 2nd order ordinary nonhomogeneous differential equation which for the case of transverse convection is linear and its analytic solution is readily available for all possible boundary conditions in any of the classic heat transfer textbooks. However, when the mode of transverse thermal radiation heat transfer is also included, the differential equation is nonlinear, and its comprehensive analytic solution still eludes us. The problem has of course been solved by numerical methods, which are quite effective, but lack sophistication and cannot easily provide the physical insights and trends when compared to analytic solutions. The necessity for such a solution is evident, especially considering that in all such applications, strictly speaking, both modes are always present.

A method by which the problem has been traditionally addressed to produce an analytic solution is the perturbation method [1]. The method has produced an analytic solution to the problem, but it is not exact, and it is subject to the condition that the nonlinear term is small such that the solution can be expanded in terms of an asymptotic series. Methods to produce a better approximate analytic solution proceeded after that, the most notable ones use a series expansion approach for the temperature distribution. These include the Homotopy Analysis Method (HAM), first introduced by Wang [2] and further developed and applied to non-linear problems by Liao [3]. The method was utilized for an extended surface with transverse radiative and convective heat transfer by Aziz and Khani [4] . Probably, the most successful technique is a derivative of the HAM referred to as the Differential Transformation Method (DTM) first introduced by Pukhov [5] and refined by Zhou [6] . The DTM was utilized successfully in an extended surface application by Dogonchi and Ganji [7] . Another notable approach is the Collocation Method used by Singh, et. al., for a convective-radiative fin with temperature dependent transport coefficients [8]. For engineering applications all these methods produce adequately accurate results, and they can be applied to the more complex extended surface problems that include temperature dependent transport properties, internal heat generation and moving fins. They are, however, somewhat cumbersome to apply due to the characteristic implementation of a series expansion for which a large number of individual terms must be developed. Furthermore, all such efforts do not address all possible boundary conditions and they all assume either a zero temperature of the surroundings, mainly when radiation is included, or equate the ambient temperature to that of the cooling fluid temperature.

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Consequently, we identify the need to generate an exact solution to the problem of heat conduction with transverse convection and radiation for all typical boundary conditions that is simple to apply. We present such an exact solution for temperature independent transport properties, and we validate by comparisons to experimental data. Since the solution is implicit, we also present an explicit approximate solution. Its utility is valuable for code verification, new approximate techniques to solve the higher-order differential equation and identification of the dependence and relevant trends of the different parameters on the temperature and heat flux profiles.

Solutions for various boundary conditions

The condition of heat conduction with transverse convective and radiative heat transfer arises frequently in various applications, the most notable ones are extended surfaces for heat rejection in thermal management of computer components, space radiators, air conditioning among many others. Figure 1 schematically represents the problem wherein heating from a source maintains a constant elevated temperature at the base of the extended surface. Heat transfer by conduction occurs along the length while convective and radiative cooling take place in the normal direction.

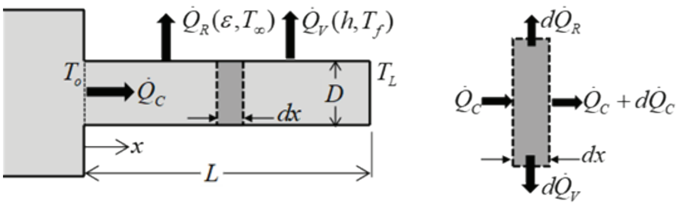

Figure 1. *Schematic of the extended surface subject to transverse cooling by radiation and convection along with the differential system for analysis*

$$\dot{Q}_{net}=0\Rightarrow \dot{Q}_C-(\dot{Q}_C+d\dot{Q}_C)-d\dot{Q}_R-d\dot{Q}_V=0\Rightarrow \quad -d\dot{Q}_C-d\dot{Q}_R-d\dot{Q}_V=0 \tag{1}$$

Fourier's law, Stefan-Boltzmann law, and Newton's law of cooling, represent conduction, radiation, and convection heat transfer rates,

$$d\dot{Q}_C=d\left(-kA_c\frac{dT}{dx}\right);\quad d\dot{Q}_R=\varepsilon\sigma\left(T^4-T_\infty^4\right)dA_s=\varepsilon\sigma P\left(T^4-T_\infty^4\right)dx;$$
$$d\dot{Q}_V=h(T-T_f)dA_s=hP(T-T_f)dx \tag{2}$$

where A_c is the cross-sectional area and P is the perimeter. Substitution into the energy conservation, imposing temperature independent transport properties, k and ε , and some rearrangement leads to the following 2nd order non-linear differential equation:

$$\frac{d^2T}{dx^2}-\frac{\varepsilon\sigma P}{kA_c}\left(T^4-T_\infty^4\right)-\frac{hP}{kA_c}(T-T_f)=0 \tag{3}$$

which can be equally applied to rectangular and cylindrical coordinates. For cylindrical geometries $P=\pi D,\ A_c=\pi D^2/4$ while for rectangular geometries $P=2(w+D),\ A_c=wD$ where *w* is the width. For most applications of interest, the boundary condition at *x = 0* is one of constant known temperature, To, thus all the different solutions that follow impose T(0)=To. However, the solutions can be easily extended to a constant heat flux or heat rate boundary condition. We reiterate that the governing differential equation is applicable to a steady-state, stationary system under the assumptions of one-dimensional conduction and temperature independent transport properties, *k, h,* and *ε*

Introducing the non-dimensional variables, $\theta=\frac{T}{T_\infty};\ \xi=\frac{x}{L}$ the differential equation with the base boundary condition of known temperature is

$$\frac{d^2\theta}{d\xi^2}=\alpha_r(\theta^4-1)+\alpha_v(\theta-\theta_f)\qquad with\quad \theta(\xi=0)=\theta_o \tag{4}$$

where $\theta_o=\frac{T_o}{T_\infty},\ \theta_f=\frac{T_f}{T_\infty},\ \alpha_r=\frac{\varepsilon\sigma P T_\infty^3 L^2}{kA_c}$ *and* $\alpha_v=\frac{hPL^2}{kA_c}$. The last two parameters are characteristic numbers depicting the ratio of radiative heat transfer to heat transfer by conduction and convective heat to heat transfer by conduction, respectively.

Solution with adiabatic boundary condition at tip; $\left.\frac{dT}{dx}\right|_{x=L}=0$

For long extended surfaces, i.e. L>>D the heat transfer rate from the cross-sectional area at the tip of the surface, $\dot{Q}_R^T=\varepsilon\sigma\frac{\pi D^2}{4}\left(T_L^4-T_\infty^4\right)$, $\dot{Q}_V^T=h\frac{\pi D^2}{4}\left(T_L-T_f\right)$ is negligible compared to the rate from the lateral surface; $\dot{Q}_R^L=\varepsilon\sigma\pi DL\left(T^4-T_\infty^4\right)$, $\dot{Q}_V^L=h\pi DL\left(T-T_f\right)$ since

$$\frac{\dot{Q}_R^L}{\dot{Q}_R^T}=\frac{4L(T^4-T_\infty^4)}{D(T_L^4-T_\infty^4)}>>1 \text{ and } \frac{\dot{Q}_V^L}{\dot{Q}_V^T}=\frac{4L(T-T_f)}{D(T_L-T_f)}>>1 \tag{5}$$

with $L>>D$ and $T>T_L$, $T_L\geq T_f$. Hence, the assumption of an insulated tip is quite reasonable and imposes an adiabatic boundary condition at $x=L$; $\left.\frac{dT}{dx}\right|_{x=L}=0\Rightarrow \left.\frac{d\theta}{d\xi}\right|_{\xi=1}=0$.

To solve the 2[nd] order, non-linear, non-homogenous differential equation we employ the following substitution: $u(\theta)=\frac{d\theta}{d\xi}\Rightarrow\frac{d^2\theta}{d\xi^2}=u\frac{du}{d\theta}$. The differential equation becomes one of separation of variables as follows:

$$u\frac{du}{d\theta}=\alpha_r(\theta^4-1)+\alpha_v(\theta-\theta_f)\ \Rightarrow\ \int_0^u udu=\int_{\theta_L}^{\theta}\left[\alpha_r(\theta^4-1)+\alpha_v(\theta-\theta_f)\right]d\theta \tag{6}$$

where the adiabatic boundary condition is imposed, $u(\theta=\theta_L)=\left.\frac{d\theta}{d\xi}\right|_{\xi=1}=0$ with $\theta_L=\frac{T_L}{T_\infty}$ which represents the non-dimensional temperature at the tip, and it is still an unknown quantity. Performing the integration and reinstating the original variables produces the following differential equation:

$$\frac{d\theta}{d\xi}=\pm\sqrt{\frac{2\alpha_r}{5}\left[\theta(\theta^4-5)-\theta_L(\theta_L^4-5)\right]+\alpha_v\left[\theta(\theta-2\theta_f)-\theta_L(\theta_L-2\theta_f)\right]} \tag{7}$$

where only the negative root represents cooling, i.e., negative temperature slope. Equation 7 represents an exact explicit expression for the heat flux since $\dot{q}(T)=-k\frac{dT}{dx}=-k\frac{T_\infty}{L}\frac{d\theta}{d\xi}$. For heating applications, the original differential equation needs to be rearranged and solved and the positive root of that solution should be selected. In this paper only cooling is addressed, but the methodology can easily be extended to the more infrequent case of heating. Separation of variables and integration produces the final implicit solution:

$$\xi(\theta)=-\int_{\theta_o}^{\theta}\frac{dt}{\sqrt{\frac{2}{5}\alpha_r\left[t(t^4-5)-\theta_L(\theta_L^4-5)\right]+\alpha_v\left[t(t-2\theta_f)-\theta_L(\theta_L-2\theta_f)\right]}} \tag{8}$$

where the base boundary condition is directly imposed. The tip non-dimensional temperature, θ_L can be iteratively obtained by the following transcendental equation since $\theta(1)=\theta_L$:

$$\int_{\theta_o}^{\theta_L}\frac{d\theta}{\sqrt{\frac{2}{5}\alpha_r\left[\theta(\theta^4-5)-\theta_L(\theta_L^4-5)\right]+\alpha_v\left[\theta(\theta-2\theta_f)-\theta_L(\theta_L-2\theta_f)\right]}}+1=0 \tag{9}$$

The heat flux at the base of the extended surface, $\dot{q}_o=-k\left.\frac{dT}{dx}\right|_{x=0}=-k\frac{T_\infty}{L}\left.\frac{d\theta}{d\xi}\right|_{\xi=0}$ can be determined utilizing (7):

$$\dot{q}_o=k\frac{T_\infty}{L}\sqrt{\frac{2}{5}\alpha_r\left[\theta_o(\theta_o^4-5)-\theta_L(\theta_L^4-5)\right]+\alpha_v\left[\theta_o(\theta_o-2\theta_f)-\theta_L(\theta_L-2\theta_f)\right]} \tag{10}$$

If the extended surface represents a means for cooling, i.e., a fin, then the performance can be assessed by the fin efficiency defined as the ratio of the actual heat transfer rate to the heat transfer rate in the absence of temperature gradients,

$$\eta_f = \frac{\dot{Q}_o}{\varepsilon\sigma A_s(T_o^4 - T_\infty^4) + hA_s(T_o - T_f)} \quad (11)$$

where $A_s = PL$ is the total radiating surface area of the fin and $\dot{Q}_o = A_c\dot{q}_o$ is the actual fin heat transfer rate. Substitution for the heat transfer rate utilizing (10) and some algebraic manipulation yields,

$$\eta_f = \frac{\sqrt{\frac{2}{5}\alpha_r\left[\theta_o(\theta_o^4 - 5) - \theta_L(\theta_L^4 - 5)\right] + \alpha_v\left[\theta_o(\theta_o - 2\theta_f) - \theta_L(\theta_L - 2\theta_f)\right]}}{\alpha_r(\theta_o^4 - 1) + a_v(\theta_o - \theta_f)} \quad (12)$$

Approximate solution

Since the exact solution is implicit, it is instructional and convenient to produce an explicit solution for the temperature, albeit approximate. The maximum ratio of heat rate due to radiative cooling to the heat rate due to convection occurs at the base,

$$q_{\max} = \left.\frac{\dot{Q}_R}{\dot{Q}_V}\right)_{\max} = \frac{\varepsilon\sigma(T_o^4 - T_\infty^4)}{h(T_o - T_f)} = \frac{\alpha_r}{\alpha_v}\left(\frac{\theta_o^4 - 1}{\theta_o - \theta_f}\right) \quad (13)$$

We note that for $q_{max} >> 1$ radiative cooling dominates, hence convective cooling can be ignored. For this case an exact solution is available and should be utilized [9]. This would occur predominantly if the base temperature is considerably larger than the ambient temperature, but for reasonable values of θ_0 we note that $\frac{\theta_o^4 - 1}{\theta_o - \theta_f} \sim O(1)$ and the ratio of heat rates effectively depends on the ratio $\frac{\alpha_r}{\alpha_v} = \frac{\varepsilon\sigma T_\infty^3}{h}$ which is always less than 1 for reasonable values of the parameters involved even if radiative cooling dominates. Hence, a rational approximation for the second derivative is a linear one as follows:

$$\frac{d^2\theta}{d\xi^2} = \theta''(\theta) = \alpha_r(\theta^4 - 1) + \alpha_v(\theta - \theta_f) \approx C_1 + C_2\theta \quad (14)$$

where derivatives will be denoted using primes. Imposing the exact boundary conditions at base and tip; $\theta''(\theta_o) = \alpha_r(\theta_o^4 - 1) + \alpha_v(\theta_o - \theta_f)$ and $\theta''(\theta_L) = \alpha_r(\theta_L^4 - 1) + \alpha_v(\theta_L - \theta_f)$ determines the constants C_1 and C_2. After some simple algebra the approximate function for the second derivative is,

$$\theta''(\theta) \approx \theta''(\theta_o) + \frac{\alpha_r(\theta_o^4 - \theta_L^4) + \alpha_v(\theta_o - \theta_L)}{\theta_o - \theta_L}(\theta - \theta_o) \quad (15)$$

where evaluation of θ_L also remains exact via Equation 9. Solution of the differential equation proceeds in the same manner as the exact approach previously outlined and leads to the following differential equation:

$$\frac{d\theta}{d\xi} = -\sqrt{C(\theta^2 - \theta_L^2) + 2[\theta''(\theta_o) - C\theta_o](\theta - \theta_L)}$$
$$\text{where } C = \frac{\alpha_r(\theta_o^4 - \theta_L^4) + \alpha_v(\theta_o - \theta_L)}{\theta_o - \theta_L} \quad (16)$$

with the boundary condition at the tip, $\left.\frac{d\theta}{d\xi}\right|_{\xi=1} = 0$ already applied. Separation of variables, integration, and implementation of the base boundary condition, $\theta(\xi = 0) = \theta_o$ yields

$$\int_{\theta_o}^{\theta} \frac{dt}{\sqrt{C(t^2 - \theta_L^2) + 2[\theta''(\theta_o) - C\theta_o](t - \theta_L)}} = -\int_0^{\xi} dy = -\xi(\theta) \quad (17)$$

The integral on the left-hand side is elementary and has a closed-form solution. Evaluation of the integral followed by some considerable algebra produces the final approximate explicit function for the non-dimensional temperature distribution. However, the approximate solution does not directly impose $\theta(1)=\theta_L$, hence an additional correction is included to enforce the condition. The final approximate solution is then

$$\theta(\xi) \approx \theta_L + \frac{\left(Be^{-\sqrt{C}\xi} - A\right)^2 + C^2\theta_L^2 + 2AC\theta_L}{2BCe^{-\sqrt{C}\xi}} - \delta \quad (18)$$

where $A = \theta''(\theta_o) - C\theta_o$, $B = \sqrt{C^2(\theta_o^2 - \theta_L^2) + 2AC(\theta_o - \theta_L)} + \theta''(\theta_o)$, $\delta = \frac{\left(Be^{-\sqrt{C}} - A\right)^2 + C^2\theta_L^2 + 2AC\theta_L}{2BCe^{-\sqrt{C}}}$ It should be noted once again that even though the temperature expression is approximate, the second derivative, $\theta''(\theta_o) = \alpha_r(\theta_o^4 - 1) + \alpha_v(\theta_o - \theta_f)$ the first derivative, $\theta'(\xi) = -\sqrt{\frac{2\alpha_r}{5}\left[\theta(\theta^4 - 5) - \theta_L(\theta_L^4 - 5)\right] + \alpha_v\left[\theta(\theta - 2\theta_f) - \theta_L(\theta_L - 2\theta_f)\right]}$ and the tip temperature, (Equation 9) are exact. Furthermore, the expressions for the heat flux (Equation 10) and efficiency (Equation 11) if a fin is the application are also exact.

To validate and perform error analysis, the solutions are compared to experimental data obtained from a simple experiment [10] utilizing a cylindrical fin cooled by radiation and natural convection. The experiment investigated three different fin configurations made of Al 2024-T4 for which the average thermal conductivity and emissivity are 120W/m/K and 0.35, respectively and are effectively independent of temperature. The authors produced a numerical solution where they determined the convective heat transfer coefficient, h based on a correlation by Churchill and Chu [11] for free convection and they allowed it to vary as a function of temperature. Including temperature variations in the calculation of h is probably unnecessary since such correlations in general do not produce accuracies better than 20%. Hence, the updated Churchill-Bernstein [12] correlation will be used for an average heat transfer coefficient applicable to cylinders subject to transverse flow and it is valid for all conditions so long as $Re_D Pr > 0.2$:

$$\bar{h} = \frac{k_f}{D}\left\{0.3 + \frac{0.62\,Re_D^{1/2}\,Pr^{1/3}}{\left[1 + (0.4/Pr)^{2/3}\right]^{1/4}}\left[1 + \left(\frac{Re_D}{282000}\right)^{5/8}\right]^{4/5}\right\} \quad (19)$$

The parameters that are temperature dependent are calculated based on the film temperature, but for this case the temperature variation is not substantial to warrant such elevated degree of accuracy. Therefore, all parameters will be calculated at STP (1atm, 300K) and they are as follows for air: and an average velocity of v=0.05m/s was assumed for natural convection. Then for all three cylindrical fins examined,   and thus the Churchill-Bernstein expression can be used. The average heat transfer coefficient calculated is presented in Table 1 along with the geometry and thermal conditions for the experiment:

Table 1. *Geometry and thermal conditions for the fins used in the experiment [3].*

FIN	D (mm)	L (cm)	$\bar{h}$(W/m²/K)	T_o (°C)	T_∞=T_f (°C)
1	3.18	68.5	14.915	92.8	20.8
2	6.35	68.5	10.054	96.0	21.0
3	9.53	90.0	8.027	89.8	20.9

The parameters above are sufficient to produce the exact solution, Equation 8, and the approximate solution, Equation 18, and since L/D>>1 the adiabatic tip boundary condition is appropriate. The maximum ratio of heat transfer by radiation to heat transfer by conduction for each fin is $q_{\max}|_{Fin1} = 0.193$, $q_{\max}|_{Fin2} = 0.291$, $q_{\max}|_{Fin3} = 0.354$, which confirms that neither radiation nor convection can be excluded as they contribute to cooling almost comparably. The non-dimensional temperature distribution along the fin is compared to the experimental measurements for all three fins in Figures 2-4. The agreement with experiment is excellent which confirms the method and the solution for both exact and the approximation. Moreover, the approximation compares very well to the exact solution. For fin 1, the maximum error produced, $err = \max\left[\frac{\theta(\xi)_{exact} - \theta(\xi)_{apprx}}{\theta(\xi)_{exact}}\right];\ 0 \le \xi \le 1$ is 0.2%, for fin 2 it is 0.3% and for fin 3 it is 0.2%. Effectively, the approximation produces negligible discrepancy for these cases. However, the conditions for this experiment resulted in a base temperature that doesn't contribute to the non-linear term substantially, . A more rigorous error analysis will be performed with the next boundary condition, constant and known tip temperature.

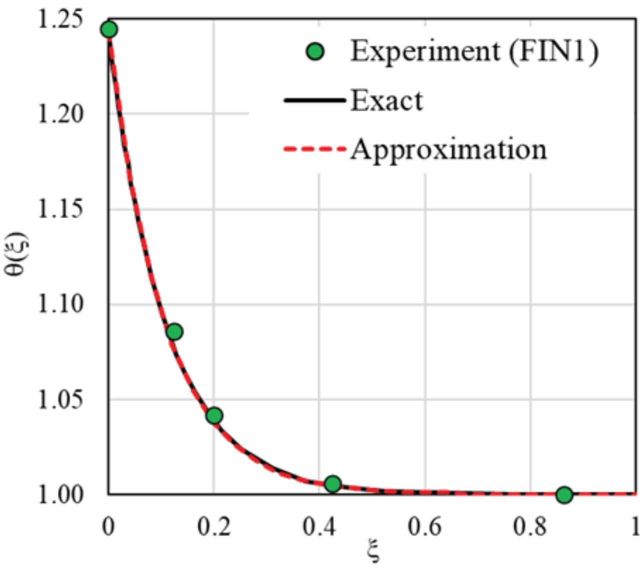

Figure 2. *Comparison of the exact (Equation 8) and approximate (Equation 18) solutions to experimental data for Fin 2*

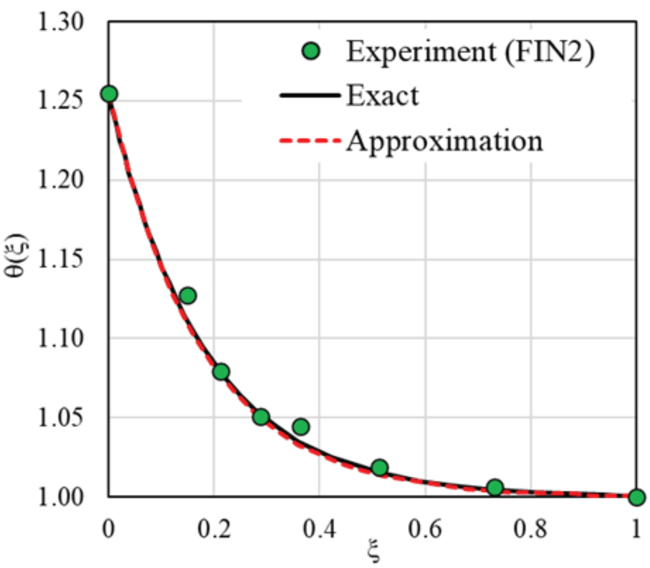

Figure 3. *Comparison of the exact (Equation 8) and approximate (Equation 18) solutions to experimental data for Fin 1*

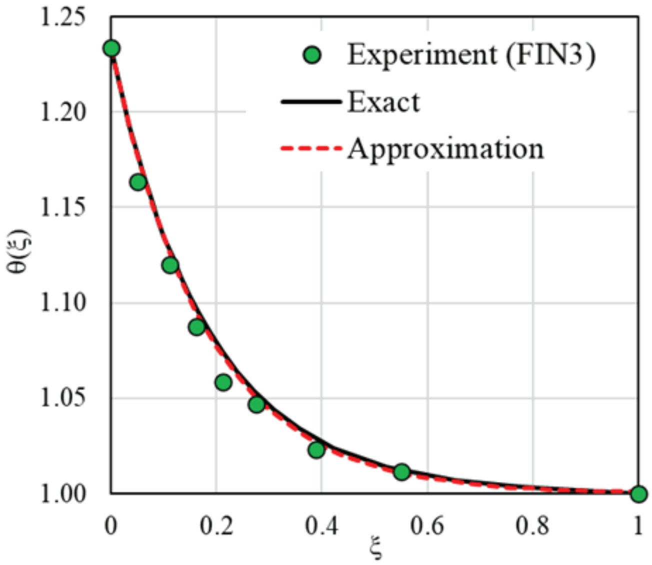

Figure 4. *Comparison of the exact (Equation 8) and approximate (Equation 18) solutions to experimental data for Fin 3*

The fin efficiency is given by

$$\eta_f = \frac{\dot{Q}_o}{\varepsilon\sigma A_s(T_o^4 - T_\infty^4) + hA_s(T_o - T_f)} \qquad (20)$$

Invoking $\dot{Q}_o = A_c\dot{q}_o = \frac{\pi D^2}{4}\left(-k\frac{T_\infty}{L}\frac{d\theta}{d\xi}\bigg|_{\xi=0}\right)$ and $A_s = PL = \pi DL,\ T_f = T_\infty$ the efficiency expression is represented in terms of the non-di mensional parameters as follows:

$$\eta_f = \frac{\sqrt{\frac{2}{5}\alpha_r\left[\theta_o(\theta_o^4 - 5) - \theta_L(\theta_L^4 - 5)\right] + \alpha_v\left[\theta_o(\theta_o - 2) - \theta_L(\theta_L - 2)\right]}}{\alpha_r(\theta_o^4 - 1) + a_v(\theta_o - 1)} \qquad (21)$$

The efficiency values of the three fins investigated are $\eta_{f1} = 0.122,\ \eta_{f2} = 0.185,\ \eta_{f3} = 0.184$.

Solution with constant temperature boundary condition at tip; $T\ (x = L) = T_L$

Starting with Equation 6,

$$u\frac{du}{d\theta} = \alpha_r(\theta^4 - 1) + \alpha_v(\theta - \theta_f) \Rightarrow \int_{u_L}^{u} u\,du = \int_{\theta_L}^{\theta}\left[\alpha_r(\theta^4 - 1) + \alpha_v(\theta - \theta_f)\right]d\theta \qquad (6)$$

where $u_L = \frac{d\theta}{d\xi}\bigg|_{\xi=1}$. Integrating and solving for $u = \frac{d\theta}{d\xi}$ we obtain

$$\frac{d\theta}{d\xi} = -\sqrt{\frac{2\alpha_r}{5}\left[\theta(\theta^4 - 5) - \theta_L(\theta_L^4 - 5)\right] + \alpha_v\left[\theta(\theta - 2\theta_f) - \theta_L(\theta_L - 2\theta_f)\right] + u_L^2} \qquad (22)$$

for cooling. Separating variables and integrating again produces the exact implicit solution:

$$\xi(\theta) = -\int_{\theta_o}^{\theta}\frac{dt}{\sqrt{\frac{2}{5}\alpha_r\left[t(t^4 - 5) - \theta_L(\theta_L^4 - 5)\right] + \alpha_v\left[t(t - 2\theta_f) - \theta_L(\theta_L - 2\theta_f)\right] + u_L^2}} \qquad (23)$$

where the non-dimensional gradient at the tip, can be obtained by the following equation:

$$\int_{\theta_o}^{\theta_L}\frac{d\theta}{\sqrt{\frac{2}{5}\alpha_r\left[\theta(\theta^4 - 5) - \theta_L(\theta_L^4 - 5)\right] + \alpha_v\left[\theta(\theta - 2\theta_f) - \theta_L(\theta_L - 2\theta_f)\right] + u_L^2}} + 1 = 0 \qquad (24)$$

The expressions for the heat flux at the base and the efficiency are determined in a similar fashion as the previous case and are as follows:

$$\dot{q}_o = k\frac{T_\infty}{L}\sqrt{\frac{2}{5}\alpha_r\left[\theta_o(\theta_o{}^4-5)-\theta_L(\theta_L^4-5)\right]+\alpha_v\left[\theta_o(\theta_o-2\theta_f)-\theta_L(\theta_L-2\theta_f)\right]+u_L^2} \quad (25)$$

$$\eta_f = \frac{A_c}{A_s}\frac{\sqrt{\frac{2}{5}\alpha_r\left[\theta_o(\theta_o{}^4-5)-\theta_L(\theta_L^4-5)\right]+\alpha_v\left[\theta_o(\theta_o-2\theta_f)-\theta_L(\theta_L-2\theta_f)\right]+u_L^2}}{\alpha_r(\theta_o^4-1)+a_v(\theta_o-\theta_f)} \quad (26)$$

where $A_s = A_c + PL$ since the tip is not insulated and thus it is subject to heat transfer by both modes.

Approximate solution

Generating the approximation proceeds in the same manner as the previous case where the second derivative is approximated by a linear function

$$\frac{d^2\theta}{d\xi^2} = \theta''(\theta) = \alpha_r(\theta^4-1)+\alpha_v(\theta-\theta_f) \approx C_1 + C_2\theta \quad (14)$$

and the constants C_1 and C_2 are evaluated at the base and tip, (see equation 15). Separation of variables (identical steps as the exact solution above) produces,

$$\frac{d\theta}{d\xi} = -\sqrt{C(\theta^2-\theta_L^2)+2[\theta''(\theta_o)-C\theta_o](\theta-\theta_L)+u_L^2}$$
$$where\ C = \frac{\alpha_r(\theta_o^4-\theta_L^4)+\alpha_v(\theta_o-\theta_L)}{\theta_o-\theta_L} \quad (27)$$

where $u_L = \left.\frac{d\theta}{d\xi}\right|_{\xi=1}$ and can be evaluated by the exact expression from equation 22. Proceeding in the same manner as the previous case, we separate variables, evaluate the elementary integrals to produce the approximate expression. A correction is also applied here to satisfy the tip boundary condition, $\theta(1) = \theta_L$ to produce the final expression,

$$\theta(\xi) \approx \theta_L + \frac{\left(Be^{-\sqrt{C}\xi}-A\right)^2 + C^2\theta_L^2 + 2AC\theta_L - Cu_L^2}{2BCe^{-\sqrt{C}\xi}} - \delta \quad (28)$$

where $A = \theta''(\theta_o) - C\theta_o,\ B = \sqrt{C^2(\theta_o^2-\theta_L^2)+2AC(\theta_o-\theta_L)+Cu_L^2} + \theta''(\theta_o)$ and $\delta = \frac{\left(Be^{-\sqrt{C}}-A\right)^2 + C^2\theta_L^2 + 2AC\theta_L - Cu_L^2}{2BCe^{-\sqrt{C}}}$

As previously, we emphasize once again that the second derivative, $\theta''(\theta) = \alpha_r(\theta^4-1)+\alpha_v(\theta-\theta_f)$, the first derivative, $\theta'(\xi) = -\sqrt{\frac{2\alpha_r}{5}\left[\theta(\theta^4-5)-\theta_L(\theta_L^4-5)\right]+\alpha_v\left[\theta(\theta-2\theta_f)-\theta_L(\theta_L-2\theta_f)\right]}$ and the tip temperature gradient, (Equation 24) are exact. Furthermore, the expressions for the heat flux (Equation 25) and fin efficiency (Equation 26) are also exact.

Error analysis allows us to establish the range that the approximate explicit expression for the non-dimensional temperature can be utilized within relative error tolerance dependent on the application and degree of accuracy required. As previously mentioned, the discrepancy between exact and approximate solutions increases as the ratio of radiative to convective heat transfer increases; $q_{\max} = \left.\frac{\dot{Q}_R}{\dot{Q}_V}\right)_{\max} = \frac{\varepsilon\sigma(T_o^4-T^4)}{h(T_o-T_f)} = \frac{\alpha_r}{\alpha_v}\left(\frac{\theta_o^4-1}{\theta_o-\theta_f}\right)$

A comparison of the approximate solution to the exact solution is presented in Figure 5 (left) for $q_{\max} = 2,\ \theta_o = 2,\ (T_o = 315^oC)\ and\ \theta_L = 1.1,\ (T_f = T_\infty = 21^oC)$. The maximum error for this case, $err = \max\left[\frac{\theta(\xi)_{exact}-\theta(\xi)_{apprx}}{\theta(\xi)_{exact}}\right];\ 0 \le \xi \le 1$ is 2.4%.

In Figure 5 (right) the maximum error is presented as a function of qmax. As expected, error increases as radiative heat transfer becomes more prevalent relative to convective heat transfer. For most applications an error of less than 6% in estimating the temperature profile is acceptable especially considering that the expressions for heat flux and cooling efficiency are exact. Furthermore, situations that result in $q_{max} > 5$ are infrequent and can be addressed by neglecting convective heat transfer. For example, for $q_{max} = 4$ using reasonable values for $\varepsilon = 0.82$, $h = 8.82\ W/m^2/K$, $T_f = T_\infty = 21°C$ the base temperature is $T_0 = 900°C$.

Applications with such infrequent high temperature values will resort to more effective means of cooling, e.g., convective cooling using liquids and/or phase change cooling. In addition, we note that a value of $h = 8.82\ W/m^2/K$ represents cooling via gas at very low speeds or simply free convection and can be neglected. In this case, simpler exact and approximate solutions [9] are available wherein convection is considered negligible. Furthermore, such elevated temperature variation along the extended surface will probably (since it is material dependent) require inclusion of temperature dependent transport properties

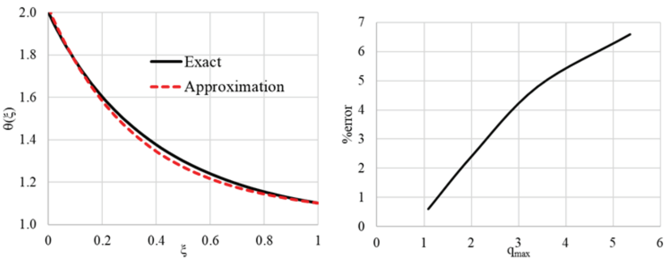

Figure 5. Left: Comparison of the exact solution (eq'n 23) to the approximate solution (eq'n 28) for $q_{max} = 2$, $\theta_o = 2$, ($T_o = 315^oC$), $\theta_L = 1.1$, $T_f = T_\infty = 21°C$. Right: Maximum error as a function of q_{max}.

Solution with radiative and convective tip boundary condition;

$$-k\left.\frac{dT}{dx}\right|_{x=L} = h(T_L-T_f)+\varepsilon\sigma\left(T_L^4-T_\infty^4\right)$$

Strictly speaking, this case represents the most general situation where there is no special treatment of or any assumptions imposed at the tip, e.g., constant temperature or insulated. Hence, it can be applied to almost any geometry with no restrictions subject only to the base boundary condition of constant known temperature, T_o. The condition of $L > D$ is still imposed to maintain the one-dimensional conduction assumption, however heat transfer from the tip can still be significant. For example, a typical case is that of computer components where maximum temperature values are of the order of 80°C, i.e., $T_o = 80^oC$. Supposing ambient and cooling fluid temperature values of 21°C, i.e., $T_f = T_\infty = 21°C$ and $T_L = 1.1\ T_\infty$ and L= 2D (4L/D = 8) the ratio of heat transfer from the lateral surface to heat transfer from the tip is

$$\frac{\dot{Q}_R^L}{\dot{Q}_R^T} = \frac{4L(T_o^4-T_\infty^4)}{D(T_L^4-T_\infty^4)} = 18.6 \text{ and } \frac{\dot{Q}_V^L}{\dot{Q}_V^T} = \frac{4L(T_o-T_f)}{D(T_L-T_f)} = 16 \quad (29)$$

These imply that about 5.4% of the total heat transfer by radiation and about 6.2% of the total heat transfer by convection, i.e., a total of more than 11% of the total heat transfer occur from the tip surface. These are minimum values since the temperature decreases along the length which implies that the average ratios calculated by (29) are actually smaller. For such cases the adiabatic tip assumption is ill-advised, and the radiative/convective tip boundary condition should be implemented.

The non-dimensional form of the radiative/convective boundary condition is

$$\left.\frac{d\theta}{d\xi}\right|_{\xi=1} = u_L = -\frac{A_c}{PL}\left[\alpha_r\left(\theta_L^4-1\right)+\alpha_v(\theta_L-\theta_f)\right] \quad (30)$$

Invoking equation 22, substitution of the expression for the boundary condition and integrating produces the exact implicit solution,

$$\xi(\theta) = -\int_{\theta_o}^{\theta} \frac{dt}{\sqrt{\frac{2}{5}\alpha_r\left[t(t^4-5)-\theta_L(\theta_L^4-5)\right]+\alpha_v\left[t(t-2\theta_f)-\theta_L(\theta_L-2\theta_f)\right]+u_L^2}} \quad (31)$$

where the tip non-dimensional temperature, can be determined by

$$\int_{\theta_o}^{\theta_L} \frac{d\theta}{\sqrt{\frac{2}{5}\alpha_r\left[\theta(\theta^4-5)-\theta_L(\theta_L^4-5)\right]+\alpha_v\left[\theta(\theta-2\theta_f)-\theta_L(\theta_L-2\theta_f)\right]+u_L^2}}+1=0 \quad (32)$$

The expressions for the heat flux at the base and the efficiency are determined in an identical fashion as the constant tip temperature case and produce the same expressions, i.e., equations 25 and 26.

$$\theta(\xi) \approx \theta_L + \frac{\left(Be^{-\sqrt{C}\xi}-A\right)^2+C^2\theta_L^2+2AC\theta_L-Cu_L^2}{2BCe^{-\sqrt{C}\xi}}-\delta \quad (33)$$

where

$$C=\frac{\alpha_r(\theta_o^4-\theta_L^4)+\alpha_v(\theta_o-\theta_L)}{\theta_o-\theta_L},\ A=\theta''(\theta_o)-C\theta_o,$$

$$B=\sqrt{C^2(\theta_o^2-\theta_L^2)+2AC(\theta_o-\theta_L)+Cu_L^2}+\theta''(\theta_o)\ \text{and}$$

$$\delta=\frac{\left(Be^{-\sqrt{C}}-A\right)^2+C^2\theta_L^2+2AC\theta_L-Cu_L^2}{2BCe^{-\sqrt{C}}}$$

which maintains exact values for the tip temperature and tip gradient. Evaluation of the approximate temperature distribution along the extended body proceeds by calculating the exact value for θ_L via equation 32 which then allows calculation of the exact value for the tip gradient, u_L via equation 30. Also as before, the expressions for the heat flux at the base and the efficiency are identical as the constant tip temperature case and produce the same expressions, i.e., equations 25 and 26. A comparison of the exact non-dimensional temperature distribution to the approximation is depicted by Figure 6 (left) for the same conditions as the constant tip temperature case. i.e., $q_{max} = 2$, $\theta_o = 2$, $(T_o = 315^oC)$, $\theta_L = 1.1$, $T_f = T_\infty = 21°C$. This comparison shows that the general case produces a slightly greater maximum error of 3.4% which is attributed to the additional presence of the quartic non-linear term in the tip gradient expression.

Figure 6 (right) presents the maximum error as a function of the maximum radiation to convection heat rate ratio, qmax which confirms the elevated error as compared to the constant tip temperature case. The maximum error produced by the approximation increases almost linearly with q_{max} and

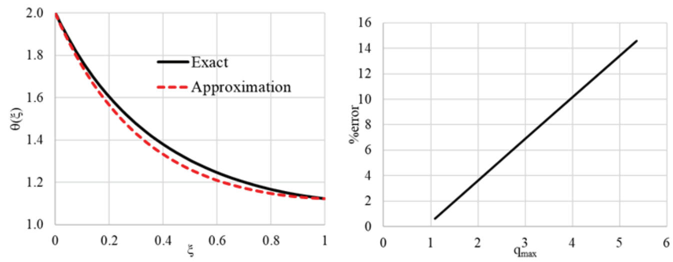

Figure 6. *Left: Comparison of the exact solution (eq'n 31) to the approximate solution (eq'n 33) for $q_{max} = 2$, $\theta_o = 2$, $(T_o = 315^oC)$, $\theta_L = 1.1$, $T_f = T_\infty = 21°C$. Right: Maximum error as a function of q_{max}.*

exceeds 7% for $q_{max} \geq 3$. Hence, if very accurate prediction of the temperature distribution in the interior of the body – base and tip temperature values are exact – is required, the approximate solution should not be used for higher values of q_{max}. Furthermore, and similar to the constant tip temperature case, values for $q_{max} > 3$ would probably require the introduction of temperature dependent transport properties.

Conclusions

Many different engineering applications present the problem of heat conduction with transverse heat transfer by convection and thermal radiation. One of the most prominent ones is utilizing such extended surfaces to enhance cooling by increasing the heat transfer area. Therefore, an exact analytic solution to the problem would be instructional in revealing the dependence of the temperature distribution on the different parameters relevant to the system and utilizing the solution for code verification or verification of more intricate versions of the same problem. Such solution has been produced for steady state, one-dimensional heat conduction with normal heat transfer by convection and radiation under the assumption of temperature independent transport properties of a stationary extended surface. The solution is generated for constant known base temperature and all possible tip boundary conditions with no assumptions on the tip, ambient and cooling fluid temperature values. It produced exact explicit expressions for the heat flux as a function of location, cooling efficiency, and exact implicit expressions for the temperature distribution and it was validated by comparison to experimental data by demonstrating excellent agreement. Because the solution for the temperature profile is implicit, an approximate explicit solution is also presented which utilizes exact expressions for heat flux and tip boundary values. The approximate solution was verified by comparisons to the exact solution and experimental data with adequate agreement to render it useful. An error analysis provided a range for which the approximate solution can be utilized with engineering acceptable discrepancy.

Nomenclature

A_c = cross-sectional area; A_s = surface area; D = diameter or thickness of extended surface; h = fluid heat transfer coefficient; k = thermal conductivity; L = length of extended surface; P = perimeter; Q = heat transfer rate; q = heat flux; q_{max} = maximum ratio of heat rate due to radiation to heat rate due to convection; T = temperature; x = dimension in the direction of heat conduction; α = non-dimensional characteristic number; ε = hemispherical emissivity; θ = non-dimensional temperature; ξ = non-dimensional dimension in the direction of heat conduction; σ = Stefan-Boltzmann constant

Subscripts

o = base; L = tip; f = fluid; ∞ = ambient, surroundings

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