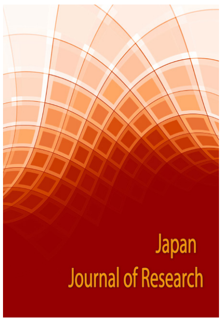
Japan Journal of Research

Original Article



Evaluation of the Accuracy of Approximations of the Differential Growth Model Using the Logistic Model

Evgueni Gordienko and Adolfo Hernández-Iglesias

Department of Mathematics, Autonomous Metropolitan University, Iztapalapa, Mexico City, Mexico.

Correspondence

Adolfo Hernández-Iglesias

Department of Mathematics, Autonomous Metropolitan University, Iztapalapa, Mexico City, Mexico.

- Received Date: 09 Aug 2025
- Accepted Date: 18 Aug 2025
- Publication Date: 29 Aug 2025

Abstract

We consider a general model of population growth given by a differential equation. Assuming that the right-hand side of the equation is unknown, we approximate the model under consideration using the classical logistic model. We establish two inequalities that evaluate the accuracy of the approximation:

(a) Upper bounds for the uniform proximity of trajectories in bounded time intervals.

(b) An upper bound for the difference between asymptotically stable states.

The results are new and original. To obtain them, we used the contractions technique, well-known in the theory of differential equations.

Introduction

Differential population growth models; Logistic model; Stable equilibrium; Contractions.

Preliminariese

In 1838, P. F. Verhulst, a Belgian biologist and mathematician, presented a model of demographic growth given by the initial value problem with the following logistic equation (1.1):.

$$\begin{cases} \dfrac{dP}{dt} = rP\left(1-\dfrac{P}{K}\right), & (1.1) \\ P(0) = P_0 > 0, & (1.2) \end{cases}$$

The system (1.1)-(1.2) aims to describe the dynamics of some population *P(t)*, $t \geq 0$, where *P(t)* can denote the number (at the time *t*) of fish, animals, bacteria, people, etc. Of course, we are talking of a territorially localized population. Also, in (1.1) *P(t)* is a differentiable function of time t which approximates the (discrete) number of members in the population.

In this model, the evolution of the population size *P(t)* is considered starting at a fixed moment t_0, which is conveniently denoted as $t_0 = 0$.

In (1.2), P_0 is a given positive starting population size. The (first order) differential equation (1.1) includes two model parameters; *r* and *K* which are given positive constants:

- $r > 0$ is called *growth rate (or coefficient)*;
- $K > 0$ is called *maximum support capacity of the environment* (or, in other terminology, *carrying capacity of the environment*).

Under the initial condition (1.2), equation (1.1) has a unique solution, given by the following formula:

$$P(t) = \frac{KP_0}{P_0 + (K - P_0)e^{-rt}}, t \in [0, \infty). \qquad (1.3)$$

Since in (1.3) $e^{-rt} \to 0$ when t→∞, we have the following asymptotic behavior of trajectories:

$$P(t) \to K \text{ when } t \to \infty,$$

that is, in this model, for all large enough times *t*, the population size *P(t)* is close to the maximum support capacity *K* of the environment.

The number *K* is the unique asymptotically stable equilibrium of equation (1.1).

Approximations and estimation of the stability (robustness) of the logistic model

The logistic model (1.1)-(1.2) is one of the basic models of population growth of a species in presence of environmental restrictions. However, there rarely are situations in which this model can be applied in its "pure form". For example, the dynamics of the American population between 1790 and 1910 were well reproduced by the logistic model (see Chapter 1 in [5]). The growth of some populations of fish follows the logistic equation (1.1). (See, for example, [1].)

Citation: Gordienko E, Hernández-Iglesias A. Evaluation of the Accuracy of Approximations of the Differential Growth Model Using the Logistic Model. Japan J Res. 2025;6(10):161.

Of course, until now many different modifications and extensions of the logistic model have appeared. Regardless, in this article we focus on the logistic model in the context of its *robustness*. We are interested in the question: to what extent can equation (1.1) work as a reasonable approximation to an unknown (for the researcher) "real" equation that describes the *growth of a certain population* $\tilde{x}(t), t \in [0,\infty]$. The actual differential equation mentioned above (see (2.2) below) may not be revealed to the researcher.

On a qualitative level, the function (changing P for x)

$$g(x) \overset{def}{=} rx\left(1-\frac{x}{K}\right),\ x \geq 0 \tag{2.1}$$

in the right side of equation (1.1) reflects well the growth of some populations, having in mind the environmental restrictions. However, for a population growth process $\tilde{x}(t)$ the parameters r and K in (2.1) may not be constant, but depend (to some extent) of the time t and the population size. For example, the carrying capacity K can vary because of technological and climatic changes. In general, in the differential equation corresponding to the "real model", the function $\tilde{g}(t,\tilde{x})$ can be different from the function g in (2.1).

Thus, we assume that the "real" trajectory $\tilde{x}(t), t \in [0,\infty]$ of the population size (in which we are interested) is the solution to the following initial value problem

$$\begin{cases} \dfrac{d\tilde{x}}{dt} = \tilde{g}(t,\tilde{x}), \quad t \geq 0 & (2.2) \\ \tilde{x}(0) = \tilde{x}_0 > 0, & (2.3) \end{cases}$$

where $\tilde{g}$ is an *unknown function* (for the researcher), for which the function g in (2.1) plays the role of an accessible approximation. The objective of the researcher is to extract useful information about the unknown trajectory of interest $\tilde{x}(t), t \in [0,\infty]$ through the analysis of the available trajectory $x(t), t \in [0,\infty)$, where (see (1.3))

$$x(t) = \frac{Kx_0}{x_0 + (K - x_0)e^{-rt}},\ t \geq 0, \tag{2.4}$$

which is the solution to the initial value problem (which we will call "approximated model"):

$$\begin{cases} \dfrac{dx}{dt} = rx\left(1-\dfrac{x}{K}\right) \equiv g(x), \quad t \geq 0 & (2.5) \\ x(0) = x_0 = \tilde{x}_0. & (2.6) \end{cases}$$

In this way, the following task of evaluation of the stability (robustness) of the model (2.5)-(2.6) arises: To estimate the closeness from $x(t)$ to $\tilde{x}(t)$ in terms of an adequate measure of the discrepancy between g and $\tilde{g}$.

Proximity of trajectories $\tilde{x}(t), t \in [0,\infty]$ and $x(t), t \in [0,\infty)$, over infinite intervals

We leave the parameters r, K in (2.5) *fixed* for now and will only consider the initial values $x_0 = \tilde{x}_0$ that belong to the interval $(0,K)$. In the first place, we have to guarantee the existence and uniqueness of the solution to the problem (2.2)-(2.3).

Assumption 2.1. The function $\tilde{g}$ en (2.2) is *continuous*, and for each $\tilde{x}_0 \in (0,K)$ there is a unique solution $\tilde{x}(t)$ to the problem (2.2)-(2.3) which is well defined for each $t \quad 0$.

Note that the local existence and uniqueness of the solution (in some neighborhood of $t = 0$) follows from the continuous differentiability of $\tilde{g}$.

In this section, we will fix a number $T > 0$ and consider the trajectories $\tilde{x}(t)$ and $x(t)$ in the interval $[0, T]$. In addition to Assumption 1, we will suppose (in this section) that

$$\tilde{x}(t) \in [0,K] \text{ for each } t \in [0,T]. \tag{2.7}$$

Using r from (2.5), we define the following constant:

$$C_T \overset{def}{=} \begin{cases} 2Te^{-2rT}, \text{ if } T < \dfrac{1}{2r}, \\ \dfrac{1}{r}e^{-1}, \text{ if } T \geq \dfrac{1}{2r}, \end{cases} \tag{2.8}$$

and introduce the following discrepancy measure between the functions $\tilde{g}(t,x)$ and $g(x)$ (see (2.2) and (2.5)):

$$\Delta(\tilde{g},g) \overset{def}{=} \max_{\substack{u\in[0,K] \\ 0\leq t\leq T}} \left|\tilde{g}(t,u) - g(u)\right|. \tag{2.9}$$

Since a continuous function reaches its maximum in closed and bounded intervals, the definition in (2.9) is consistent and $\Delta(g,g)$ is a finite number. The *stability* (or robustness) *inequality* (2.10) given below states that if the (unknown) function $\tilde{g}(t,x)$ in (2.2) is *uniformly* close to the function $g(x)$ in (2.5), then the *maximum deviation* of $\tilde{x}(t)$ with respect to $x(t)$ is also *small*. Consequently, the logistic trajectory $x(t), t \in [0,T]$ given in (2.4) can be used to approximate (and analyze) the ("real") trajectory $\tilde{x}(t), t \in [0,T]$ of the population in question. This same inequality (2.10) allows us to estimate the error of the approximation.

The proof of the below Theorem 2.1 uses the technique presented in Chapter 3 in [2].

Theorem 2.1. *Suppose that* $\tilde{x}_0 = x_0 \in (0,K)$. *Under Assumption 2.1 and (2.7), the following inequality holds:*

$$\max_{0\leq t\leq T}\left(e^{-2rt}\left|\tilde{x}(t) - x(t)\right|\right) \leq C_T \Delta(\tilde{g},g), \tag{2.10}$$

where C_T *and* $\Delta(\tilde{g},g)$ *are defined in (2.8) and (2.9).*

Let $X \equiv \mathbb{C}[0,T]$ be the space of all continuous functions $z(t), t \in [0,T]$. We equip X with the following *metric*: for $z = z(t), y = y(t) \in X$, let

$$d(z,y) \overset{def}{=} \max_{0\leq t\leq T}\left(e^{-2rt}\left|z(t) - y(t)\right|\right). \tag{2.11}$$

We define two operators T and $\tilde{T}: X \to X$. For $z = z(t) \in X$ let:

$$Tz[t] \overset{def}{=} x_0 + \int_0^t g(z(s))ds,\ t \in [0,T], \tag{2.12}$$

$$\tilde{T}z[t] \overset{def}{=} x_0 + \int_0^t \tilde{g}(s,z(s))ds,\ t \in [0,T], \tag{2.13}$$

where the function g was defined in (2.1) and $\tilde{g}$ is the right side of equation (2.2).

We will verify that if $\tilde{x} = \tilde{x}(t)$ is the solution to the problem (2.2)-(2.3), then

$$\tilde{x} = \tilde{T}\tilde{x}, \tag{2.14}$$

or

$\tilde{T}\tilde{x}[t] = \tilde{x}(t)$ for each $t \in [0,T]$. In fact, rewriting (2.14) as

$$\tilde{x}(t) = x_0 + \int_0^t \tilde{g}(s,\tilde{x}(s))ds,\ t \in [0,T],$$

and differentiating this inequality (using the continuity of $\tilde{g}$), we get

$$\frac{d\tilde{x}(t)}{dt} = \tilde{g}(t,\tilde{x}(t)),\ t \in [0,T],$$

which is equation (2.2).

Similarly, if $x = x(t)$ is the solution to the problem (2.5)-(2.6), then

$$x = Tx \tag{2.15}$$

(see (2.12)).

In other words, the functions $\tilde{x} = \tilde{x}(t)$ and $x = x(t)$ are fixed points of the operators $\tilde{T}$ and T, respectively.

The next step is to verify the inequality (2.16) below: if $z = z(t)$, $y = y(t)$ are elements of X such that $z(t) \in [0, K]$; $y(t) \in [0, K]$, $t \in [0, T]$, then

$$d(Tz, Ty) \le 0.5 d(z, y). \tag{2.16}$$

First, we verify that for all numbers $u, u' \in [0, K]$,

$$|g(u) - g(u')| \le r|u - u'|. \tag{2.17}$$

Indeed, (see (2.1)), $g'(u) = r\left(1 - \frac{2u}{K}\right)$ and the maximum value of $|g'(u)|$ in $[0, K]$ is r.

Then, (2.17) follows from the inequality:

$$|g(u) - g(u')| \le \sup_{c \in [0,K]} |g'(c)||u - u'|$$

Then, by (2.12) and (2.17), for each $t \in [0, T]$

$$\begin{aligned} |Tz[t] - Ty[t]| &= \left| \int_0^t g(z(s))\,ds - \int_0^t g(y(s))\,ds \right| \\ &\le \int_0^t |g(z(s)) - g(y(s))|\,ds \\ &\le r\int_0^t |z(s) - y(s)|\,ds \\ &= r\int_0^t e^{-2rs} e^{-2rs} |z(s) - y(s)|\,ds \end{aligned}$$

$$\le \text{see } (2.11) \le r\int_0^t e^{2rs} d(z, y)\,ds. \tag{2.18}$$

But

$$\int_0^t e^{2rs}\,ds = \frac{1}{2r}\left(e^{2rt} - 1\right) < \frac{1}{2r} e^{2rt}.$$

The combination of (2.18) and (2.19) implies that $|Tz[t] - Ty[t]| < 0.5 d(z, y) e^{2rt}$. Dividing the last equation by e^{2rt}, we get that $\sup_{0 \le t \le T} e^{-2rt} |Tz[t] - Ty[t]| \le 0.5 d(z, y)$ or, in view of (2.11), $d(Tz, Ty) \le 0.5 d(z, y)$

According to (2.11), to prove (2.10) we need to find an upper bound for $d(x, \tilde{x})$. Using (2.14), (2.15), (2.7), (2.16) and the triangle inequality for the metric d, we get:

$$d(x, \tilde{x}) = d(Tx, \tilde{T}\tilde{x}) \le$$
$$d(Tx, T\tilde{x}) + d(T\tilde{x}, \tilde{T}\tilde{x}) \le$$
$$0.5 d(x, \tilde{x}) + d(T\tilde{x}, \tilde{T}\tilde{x}), \text{or}$$
$$d(x, \tilde{x}) \le 2d(T\tilde{x}, \tilde{T}\tilde{x}). \tag{2.20}$$

Applying (2.11), (2.12) and (2.13), we obtain that

$$d(T\tilde{x}, \tilde{T}\tilde{x}) = \max_{0 \le t \le T} \left\{ e^{-2rt} \left| \int_0^t g(\tilde{x}(s))\,ds - \int_0^t \tilde{g}(s, \tilde{x}(s))\,ds \right| \right\}. \tag{2.21}$$

Then, using (2.9) and (2.7),

$$\left| \int_0^t g(\tilde{x}(s))\,ds - \int_0^t \tilde{g}(s, \tilde{x}(s))\,ds \right| \le \int_0^t |g(\tilde{x}(s)) - \tilde{g}(s, \tilde{x}(s))|\,ds$$
$$\le \int_0^t \Delta(\tilde{g}, g)\,ds = t\Delta(\tilde{g}, g). \tag{2.22}$$

From (2.21) and (2.22) it follows that

$$d(T\tilde{x}, \tilde{T}\tilde{x}) \le \Delta(\tilde{g}, g) \max_{0 \le t \le T} t e^{-2rt}. \tag{2.23}$$

When differentiating, we easily see that the function $h(t) \overset{\text{def}}{=} te^{-2rt}$, $t \ge 0$ takes its maximum at $t_* = \frac{1}{2r}$, and also is increasing in $[0, t_*]$.

Then, $\max_{0 \le t \le T} h(t) = h(t_*) = \frac{1}{2}\frac{1}{r}e^{-1}$ if $T \ge \frac{1}{2r}$ and $\max_{0 \le t \le T} h(t) = h(T) = Te^{-2rt}$ if $T \ge \frac{1}{2r}$.

Taking into account the definition of C_T in (2.8), in both cases,

$$\max_{0 \le t \le T} h(t) \le \frac{1}{2} C_T. \tag{2.24}$$

Joining inequalities (2.20), (2.23) and (2.24), we get the desired inequality (2.10).

Example 2.1. Let in (2.2) $\tilde{g}(t, \tilde{x}) = \tilde{r}(t, \tilde{x})\tilde{x}\left(1 - \frac{\tilde{x}}{K}\right)$, , that is, the "real model" (2.2)-(2.3) is given by a differential equation similar to (2.5), but instead of the constant growth rate r, we admit that this coefficient can depend on the size of the population $\tilde{x}$ and the time t.

We suppose that the function $\tilde{r}(t, \tilde{x})$ is such that Assumption 2.1 and inequality (2.7) hold, and for a "small" given $\varepsilon > 0$,

$$\max_{\substack{\tilde{x} \in [0,K] \\ 0 \le t \le T}} |\tilde{r}(t, \tilde{x}) - r| \le \varepsilon, \tag{2.25}$$

where the constant r appears in (2.4), (2.5).

According to (2.9),

$$\Delta(\tilde{g}, g) = \max_{\substack{u \in [0,K] \\ 0 \le t \le T}} \left| (\tilde{r}(t, u) - r)\left(u - \frac{u^2}{K}\right) \right| \le \varepsilon \max_{u \in [0,K]} \left| u - \frac{u^2}{K} \right| = \varepsilon \frac{K}{4}.$$

Then, $\Delta(\tilde{g}, g) \le \varepsilon \frac{K}{4}$ and by (2.10),

$$\max_{0 \le t \le T} e^{-2rt} |\tilde{x}(t) - x(t)| \le C_T \frac{K}{4} \varepsilon. \tag{2.26}$$

We consider, for example, the following numeric data:

- $T = 5$ (years);
- $K = 1$ (a million animals);
- $r = 0.1$;
- $\tilde{x}_0 = x_0 = 0.2$ (millions)
- Also, let in (2.25) $\varepsilon = 0.01$.

Using the "approximated model" (2.5)-(2.6), by (2.4) we get:

$$x(5) = \frac{0.2}{0.2 + 0.8e^{-0.5}} \approx 0.2919$$
$$(= 291.9 \text{ thousand animals})$$

On the other hand, in (2.8), $T = 1 / 2r = 5$, and

$$C_T = \frac{1}{r} e^{-1} = 10e^{-1} < 3.7$$

Then, in this example, the right side of inequality (2.10) is less than $3.7 \cdot \frac{1}{4} 0.01 = 0.00925$ (see (2.26)). According to (2.10),

$$|\tilde{x}(5) - x(5)| \le e^{2rT} \cdot 0.00925 < 0.02514.$$

Therefore, the error of the estimation of the unknown value $\tilde{x}(5)$, using the approximation by the logistic model (2.5)-(2.6), is less than 25.14 thousand animals, that is, $\tilde{x}(5)\in[266.76,317.04]$ (in thousands of animals).

Observation 2.1. The simple method used above to prove the inequality (2.10) also works for differential equations with control, and even for controlled stochastic equations (see [3]).

Robustness of the asymptotic behavior of the logistic model

Let us rewrite equation (2.5) using the notations y and τ for the corresponding variables (instead of x and t in (2.5)):

$$\frac{dy}{d\tau}=ry\left(1-\frac{y}{K}\right). \qquad (2.27)$$

For the analysis of the qualitative properties of its solution $y(\tau)$, in particular the fact that:

$$\lim_{\tau\to\infty} y(\tau)=K, \qquad (2.28)$$

without loss of generality we can suppose that

$$r = K = 1 \qquad (2.29)$$

Indeed, as it is easy to verify, with the change of variables $t = r\tau$ and $x=\frac{y}{K}$, equation (2.27) becomes the following particular logistic equation (with $r = K = 1$):

$$\frac{dx}{dt}=x(1-x). \qquad (2.30)$$

Fixing an initial value $x_0\in(0,1)$, the only solution of (2.30) is the following strictly increasing function.

$$x(t)=\frac{x_0}{x_0+(1-x_0)e^{-t}},\ t\ge 0, \qquad (2.31)$$

such that

$$\lim_{t\to\infty} x(t)=x_*\overset{\text{def}}{=}1. \qquad (2.32)$$

Considering in what follows that all equations have variables for which (2.29) is valid, we go back to the "real population" model (2.2)-(2.3), but now we assume that:

a the function $\tilde{g}$ in (2.2) does not depend on the time *t*;

b the initial value problem given in (2.33) has a unique solution $\tilde{x}(t)$, which is well defined for all $t \ge 0$. Here, we deal with the initial value problem:

$$\begin{cases}\dfrac{d\tilde{x}}{dt}=\tilde{g}(\tilde{x}),\\ \tilde{x}(0)=\tilde{x}_0,\end{cases} \qquad (2.33)$$

where, as before, the function $x - x^2$ in (2.30) works as an available approximation for the unknown function $\tilde{g}$ in (2.33).

In many situations, the most important thing to know is not the proximity of $\tilde{x}(t)$ (the solution of (2.33)) to $x(t)$ (the solution of (2.30) with $x_0=\tilde{x}_0$ over short periods of time, but the answers to the following questions:

1. When does the following limit exist:

$$\lim_{t\to\infty}\tilde{x}(t)=\tilde{x}_*\,? \qquad (2.34)$$

2. In what conditions is $\tilde{x}_*$ close to $x_* = 1$ (see (2.32))?

In other words, we are interested in *the affinity of the asymptotic behavior* of the "approximated" and "real" models.

3. Under what conditions is the "real" trajectory $\tilde{x}(t)$, $t\in[0,\infty]$ a strictly increasing function of time?

To give reasonable answers to these questions, we need the following assumption.

Assumption 2.2.

(a) The function $\tilde{g}$ in (2.33) is differentiable.

(b) $\tilde{g}(0)=0$ and $\tilde{g}'(0)>0$

(c) In (0, ∞), the algebraic equation $\tilde{g}(x)=0$ has a *unique* root $\tilde{x}_*$ which belongs to the interval [0.8,1.2], that is,

$$\tilde{x}_*\in[0.8,1.2].$$

(d) For each $x\in[0.8,1.2], \tilde{g}'(x)<0$

Observation 2.2.

a The interval [0.8,1.2] appears because we will show that $\lim_{t\to\infty}\tilde{x}(t)=\tilde{x}_*$, and evaluate the closeness of $\tilde{x}_*$ to $x_* = 1$ in (2.32).

b Assumption 2.2, (b) could be changed for $\tilde{g}(x_0)>0$.

Theorem 2.2. *Suppose that* $\tilde{x}_0=x_0\in\left(0,\min\{1,\tilde{x}_*\}\right)$. *Under Assumption 2.2, we have that:*

- The solution $\tilde{x}(t), t\in[0,\infty)$ of the problem (2.33) is a *strictly increasing* function of time, and $\tilde{x}(t)<\tilde{x}, t\ge 0$.
- $$\lim_{t\to\infty}\tilde{x}(t)=\tilde{x}_*. \qquad (2.35)$$
- $$|1-\tilde{x}_*|\le\min\left\{0.2,\ 2.27\max_{0.8\le x\le 1.2}\left|\sqrt{x}-\tilde{f}^{-1}(x)\right|\right\}, \qquad (2.36)$$

where $\tilde{f}^{-1}(x)$ is the inverse function of the following function

$$f(x)\overset{\text{def}}{=}x-g(x), x\in[0.8,1.2]. \qquad (2.37)$$

Proof. In the first place, we observe that in view of Assumption 2.2, (d), in the interval [0.8,1.2] the inverse function $\tilde{f}^{-1}$ of the function $\tilde{f}$ in (2.37) is well defined. In fact, $\tilde{f}'(x)=1-\tilde{g}'(x)>1$, and the function $\tilde{f}$ is strictly increasing.

Assumption 2.2, (b) and (c) tells us that the points and are equilibrium states of the autonomous equation

$$\frac{d\tilde{x}}{dt}=\tilde{g}(\tilde{x})\equiv\tilde{x}-\tilde{f}(\tilde{x})$$

(see (2.37)). Also (see (b) and (d) of Assumption 2.2 and, for example, the book [4]), the equilibrium $\tilde{x}'=0$ is unstable, but the equilibrium $\tilde{x}_*\in[0.8,1.2]$ is *asymptotically stable.*

It is well-known (consult, for example, [6]) that every trajectory of an *autonomous* first order differential equation is a strictly monotonic function (as long as x_0 is not an equilibrium). On the other hand, the trajectory cannot approximate or cross the level corresponding to an unstable equilibrium (see [6]). Then, $\tilde{x}(t)$ should increase strictly, and since $\tilde{x}(t)<\tilde{x}_*, t\ge 0$ it must be that $\tilde{x}(t)\to\tilde{x}_*$ when $t\to\infty$.

To obtain the inequality (2.36), we see that the point $x_*=1$ is the solution of the equation $x = x^2$ (see (2.30)), or $x=\sqrt{x}$ According to Assumption 2.2, (c) and (2.38), $\tilde{x}_*=\tilde{f}(\tilde{x}_*)$, or $\tilde{x}_*=\tilde{f}^{-1}(\tilde{x}_*)$. Then,

$$\begin{aligned}|1-\tilde{x}_*|\equiv|x_*-\tilde{x}_*|&=\left|\sqrt{1}-\tilde{f}^{-1}(\tilde{x}_*)\right|\\ &\le\left|\sqrt{1}-\sqrt{\tilde{x}_*}\right|+\left|\sqrt{\tilde{x}_*}-\tilde{f}^{-1}(\tilde{x}_*)\right|\\ &\le\max_{0.8\le x\le 1.2}(\sqrt{x})'\,|1-\tilde{x}_*|+\left|\sqrt{\tilde{x}_*}-\tilde{f}^{-1}(\tilde{x}_*)\right|\\ &\le 0.5591|1-\tilde{x}_*|+\max_{0.8\le x\le 1.2}\left|\sqrt{x}-\tilde{f}^{-1}(x)\right|.\end{aligned}$$

Subtracting the first term, we get:

$$\left|1-\tilde{x}_*\right| \le 2.269 \max_{0.8\le x\le 1.2}\left|\sqrt{x}-\tilde{f}^{-1}(x)\right|.$$

Example 2.2.

a For some $\varepsilon \in [0,0.1]$, let us consider the case when in (2.33) $\tilde{g}(x)=(1+\varepsilon)x-x^2$. It is easy to see that Assumption 2.2 holds in this case.

The equation $(1+\varepsilon)x-x^2=0$ leads us to $\tilde{x}_*=1+\varepsilon$. Then, the left side of inequality (2.36) is equal to ε.

On the other hand, we will estimate the right side of (2.36). By (2.37), $\tilde{f}(x)=x-\left[(1+\varepsilon)x-x^2\right]=-\varepsilon x+x^2$. Let $z \in [0.8,1.2]$. To find $\tilde{f}^{-1}(z)$ we have to solve the quadratic equation

$$-\varepsilon x+x^2=z, \text{ or}$$

$$x^2-\varepsilon x-z=0, \text{whose positive root is}$$

$$\tilde{f}^{-1}(z)=\frac{\varepsilon}{2}+\sqrt{\frac{\varepsilon^2}{4}+z}>\sqrt{z}.$$

Since $\sqrt{z+\delta}=\sqrt{z}+\frac{1}{2}\frac{1}{\sqrt{z}}\delta+\dots$, the maximum (in [0.8,1.2]) of $\tilde{f}^{-1}(z)-\sqrt{z}$ is reached for $z = 0.8$. Therefore, the second term in the right side of (2.36) is

$$2.27\left(\frac{\varepsilon}{2}+\sqrt{0.8+\frac{\varepsilon^2}{4}}-\sqrt{0.8}\right). \quad (2.39)$$

For example, for $\varepsilon = 0.1$, the expression in (2.39) is less than 0.1167. Having in mind that the left side of (2.36) is 0.1, we can conclude that in this example the "stability inequality" (2.36) works perfectly (that is, it is quite precise).

Example 2.3. For a given $\varepsilon \in [0,0.1]$, let in (2.33), (2.38) $\tilde{g}(x)=x-x^2-\varepsilon x^4$, or in (2.37) $\tilde{f}(x)=x^2+\varepsilon x^4$. To find the inverse $\tilde{f}^{-1}(z)$ we have to solve the equation $x^2+\varepsilon x^4=z$, or with $y = x^2$, $y + \varepsilon y^2 = z$, or $\varepsilon y^2 + y - z = 0$. The positive root is $y_*=\frac{-1+\sqrt{1+4\varepsilon z}}{2\varepsilon}$. Therefore,

$$\tilde{f}^{-1}(z)=\left(\frac{-1+\sqrt{1+4\varepsilon z}}{2\varepsilon}\right)^{1/2}. \quad (2.40)$$

Using (2.40), it is not difficult to verify that the function $\sqrt{z}-\tilde{f}^{-1}(z)$ is positive and increasing in the interval [0.8,1.2]. Therefore, the maximum in (2.36) is reached in the point $x = 1.2$.

Suppose that $\varepsilon = 0.1$. With a little arithmetic, we find that the right side of the inequality (2.36) is less than **0.1246.**

To find the value of the equilibrium $\tilde{x}_*$, we need to look for the positive (and real) root of the equation: $\tilde{g}(x)=0$, or $x-x^2-\varepsilon x^4=0$, or $1-x-\varepsilon x^3=0$. Let $\varepsilon = 0.1$. Using the cubic equation solution calculator (in Google), we get that:

$$\tilde{x}_* \approx 0.9217.$$

Then, the left side of (2.36) is equal to **0.0783**. We see that in this case the difference between the right and left sides of the inequality (2.36) is 0.0463.

In any case, this is half of the $\varepsilon = 0.1$ (which is used to compare $\tilde{g}(x)=x-x^2-\varepsilon x^4$ with $x - x^2$ in (2.30)).

Example 2.2.

b In the formulation of Example 2.2, we will try applying the stability inequality (2.10) given in Theorem 2.1.

Let, for example, $\varepsilon = 0.1$ and $T = 0.5$. Then in (2.9) (since $r = 1$ and $K = 1$),

$$\max_{x\in[0,1]}\left|(1+\varepsilon)x-x^2-(x-x^2)\right|=\varepsilon.$$

In (2.8) $C_T=e^{-1}$. Applying (2.10), we get that

$$\max_{0\le t\le 0.5}\left|\tilde{x}(t)-x(t)\right|\le e^{2\cdot 0.5}e^{-1}\varepsilon=\varepsilon=0.1.$$

Observation 2.3. It is worth recalling that, in general, an asymptotically stable equilibrium *is not robust* to small changes in the parameters of the differential equation. One simpler example is the equation

$$\frac{dy}{dt}=\alpha y\left(\text{with } y(0)=y_0>0\right).$$

Here, for an arbitrarily small $\varepsilon > 0$, $y(t) \to 0$ if $\alpha = -\varepsilon$ and $y(t) \to \infty$ if $\alpha = \varepsilon$ (when $t \to \infty$).

References

1. Abakumov, and Y. G. Izrailsky, "The Harvesting Effect on a Fish Population," Matem. Biol. Bioinf., vol. 11, no. 2, pp. 191-204, September 2016
2. R. E. Edwards, Functional Analysis. Theory and Applications. New York: Dover Publications, 1995.
3. E. Gordienko, and E. Lemus-Rodríguez, "Estimation of robustness for controlled diffusion processes,". Stochastic Anal. Appl., vol. 17, no. 3, pp. 421-441, April 1999.
4. J. D. Logan, A First Course in Differential Equations, 2nd. ed. New York: Springer, 2011.
5. J. D. Murray, Mathematical Biology, Vol. I: An Introduction, 3rd. ed. New York: Springer, 2002.
6. P. J. Olver. (2022, July 1st). Nonlinear Ordinary Differential Equations [Online]. Available: https://www-users.cse.umn.edu/~olver/ln_/odq.pdf