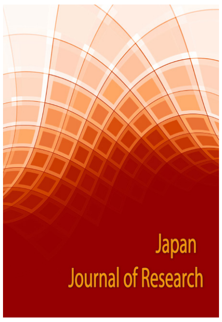
Japan Journal of Research

Original Article



Visualization of Potential Flow: Dealing with the Branch Cuts

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- Received Date: 18 July 2024
- Accepted Date: 27 July 2024
- Publication Date: 30 July 2024

Abstract

The theory of potential flow is important in several application fields, in particular: in aero dynamics, fluid dynamics, electrostatics, porous media flow. In the classical approach flow is described by a potential function of the complex plane into the complex plane. Using potential function contouring the visualizing of streamlines in flow nets can reveal important characteristics of the flow in question, for which alternative methods fail. However, branch cuts of important potential functions pose a problem for the visual representation. Here methods are outlined how to deal with the problem: (1) to be aware that equalities, common in real number algebra, may not hold in the complex number space; (2) to partition the complex plane into sub-domains on which the functions are evaluated sequentially. The description of the methods uses superpositions of complex logarithms as examples, but the ideas can be adopted for the visualisation to other potentials as well.

Introduction

Potential flow as referred to in current terminology is a classical topic. It's basic formulation goes back to the first formulations of fluid flow by Leonard Euler in 1752 [1]. It refers to a flow field with the velocity v given by the negative gradient of a function, which is called the potential function ϕ :

$$\mathbf{v} = -\nabla\phi \qquad (1)$$

Euler dealt with three-dimensional flow real valued functions ϕ. The theory of flow described by equation (1) gained much higher importance, when dealing with flow in two dimensions and complex valued potential functions $\Psi = \phi + i\psi$ [2,3]. The real part ϕ of the complex potential Ψ is referred to as the real potential; the imaginary part ψ is called the streamfunction. The naming of ψ refers to its important property: Its contour lines are streamlines of the flow field. It is this property what makes the complex potential so important and convenient for visualisation.

Most software and programs for flow-net visualisation use particle tracing as tool for streamline visualisation [4], while the alternative method of streamfunction contouring is seldom applied. This is unfortunate because using the streamfunction has two important advantages: (1) it is exact, while in particle tracing errors performed in each time step accumulate [5]; (2) if the streamfunction levels for contouring are chosen equidistantly the flow between adjacent streamlines is the same. The streamtubes are equal. The latter property can be utilized for visual flux balancing (see Figure 7 and description as an example). Particle tracing depends on the choice of starting points, which is usually done manually by the user. Thus it is not guaranteed that there is equal flow in streamtubes between streamlines.

However, there are problems in the straightforward application of this property. These are outlined here using the complex logarithm as example. Figure 1 depicts two visualisations of the complex logarithm $\Psi = \log(z)$ with the complex variable $z = x + iy$.

The left figure uses a black and white colour map for the real potential and white contour lines for the streamfunction. It has to be noted here that the streamline on the negative real axis is thicker than the others. The right figure uses the colour map for the streamfunction and contour lines for the real potential. It indicates much more clearly that there might be a problem with the streamfunction at the negative axis: values of ψ change from the lowest value (white colour) to the highest (black colour).

Citation: Holzbecher E. Visualization of Potential Flow: Dealing with the Branch Cuts. Japan J Res. 2024;5(6):45.

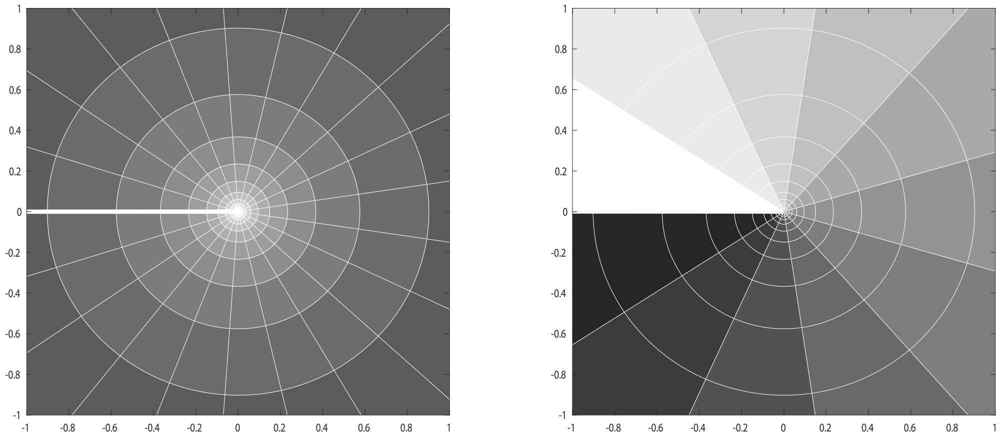

***Figure 1**. Flow-net visualizations of the Complex Logarithm*

The jump results from the definition of the complex logarithm:

$$log\left(z\right) = log(re^{i\varphi}) = log\left(r\right) + i\varphi \qquad (2)$$

where (r,φ) are the polar coordinates in the complex plane. As the angle φ is not unique, the complex logarithm is in fact a multi-valued function. For all whole numbers n with $\phi + 2\pi n$ the logarithm (2) maps to the same complex number. In computer languages the implemented log function is made single valued by choosing φ in the interval $]-\pi,\pi]$, which is referred to as the principal value or principal branch of the logarithm [6]. Evidently the jump of the complex potential at the negative real axis is $2\pi i$.

Despite the mentioned discontinuity the flow-nets in Figure 1 are correct. However, the inconsistency becomes more serious visually, when two logarithms are added:

$$\Psi\left(z\right) = log(z - \frac{1}{2}i) + log(z + \frac{1}{2}i) \qquad (3)$$

Such superposition is an important technique that is valid and extensively applied in potential flow. Figure 2 uses the same visualisation as outlined before for the function given by equation (3)

The flow-net on the left of Figure 2 has obviously two problems: (1) the thick white lines that extend from the locations of the logarithms to the left are not streamlines; (2) streamlines that reach the two thick lines from top and from bottom do not match. The right figure reveals changes of the streamfunction from lowest to highest values (analogous to what was observed in the right of Figure 1) exactly at the two mentioned lines. A method to overcome this problem is outlined here.

The importance of potential flow stems from its relation to complex analysis. Using equation (1) in the continuity equation $\nabla \cdot \mathbf{v} = 0$ as law of volume conservation one obtains the Laplace equations $\nabla^2\phi = 0$ and $\nabla^2\psi = 0$ as well as the Cauchy-Riemann Equations

$$\frac{\partial\phi}{\partial x} = \frac{\partial\Psi}{\partial y} \qquad \frac{\partial\phi}{\partial y} = -\frac{\partial\Psi}{\partial x} \qquad (4)$$

The Cauchy-Riemann equations are fundamental in the field of complex analysis [2]. In the following we utilize the complex logarithm to exemplify the derived methodology. In fact in potential flow applications the logarithm is the one of most important functions, at it is used to describe a line sink or source:

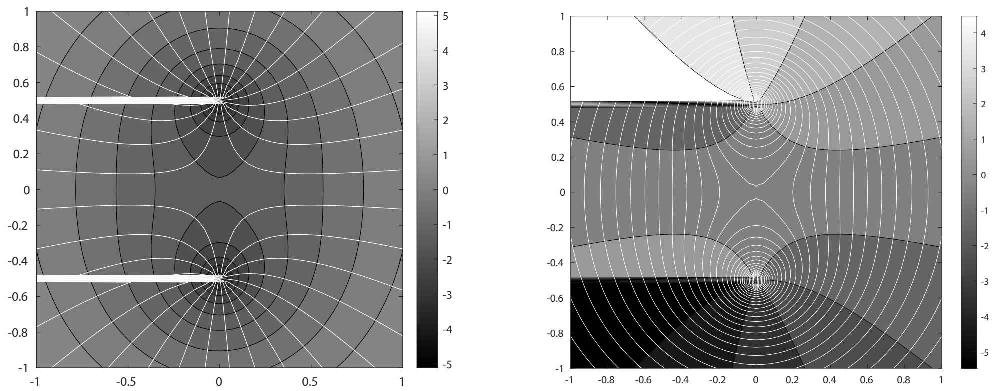

***Figure 2.** Flow-net visualisations for the superposition of two complex logarithms*

$$\Psi\left(z\right)=\frac{Q}{2\pi}\log(z-z_0) \qquad (5)$$

where Q denotes the sink- and source-rate and z_0 its location in the complex plane. The logarithm also describes a line vortex:

$$\Psi(z)=\frac{i\Gamma}{2\pi}\log(z-z_0) \qquad (6)$$

where Γ denotes the vertex circulation. For the vortex the branch cut appears in the real part of the complex potential, while the stream function does not show problems with the visualisation. In the following we restrict ourselves to examples, in which the coefficient of the logarithm is real valued as in equation (5).

Methods

Identities in the Complex Plane

Some identities that one is used to dealing with real numbers do not hold, when the number space is extended to the complex plane. $\sqrt{z^2}=z$ for example does not hold for $z=-1$. $log\left(1/z\right)=-log\left(z\right)$ is not true at the negative real axis. The reason for this are the branch cuts. In all implementations of computational algebra the branch cut of the complex logarithm is located along the negative real axis, as demonstrated in Figure 1 using MATLAB® [7].

This has consequences for the visualisation of complex functions, as will be demonstrated here for the identity

$$log(z_1)+log(z_2)=log(z_1z_2) \qquad (7)$$

which is valid on the positive real axis. As an example we compare both sides of (7) this for log (-i):

$$\begin{aligned} &log((-1)i)=log(-i)=-\frac{1}{2}\pi \\ &log(-1)+log\left(i\right)=\pi+\frac{1}{2}\pi=\frac{3}{2}\pi \end{aligned} \qquad (8)$$

In order to find out, what that means for the visualization of complex potentials we will examine functions that include the complex logarithm. Lets start comparing the functions

$$\begin{aligned} &\Psi_1\left(z\right)=log(z-z_1)+log(z-z_2) \\ &\Psi_2\left(z\right)=log\left[(z-z_1)(z-z_2)\right] \end{aligned} \qquad (9)$$

with $z_1=1/2i$ and $z_2=-1/2i$. For real values Ψ_1 and Ψ_2 coincide. The resulting flow nets are depicted in Figure 3. The two figures on the left visualize Ψ_1, the figures on the right Ψ_2. The upper figures use black and white colour maps for the real potential and white contour lines for the streamfunction. The graphics for Ψ_1 and Ψ_2 show the same flow pattern. However, all four visualisations have the above-mentioned problems at branch cuts: not matching streamlines in the two upper sub-figures and jumps from high to low values of the

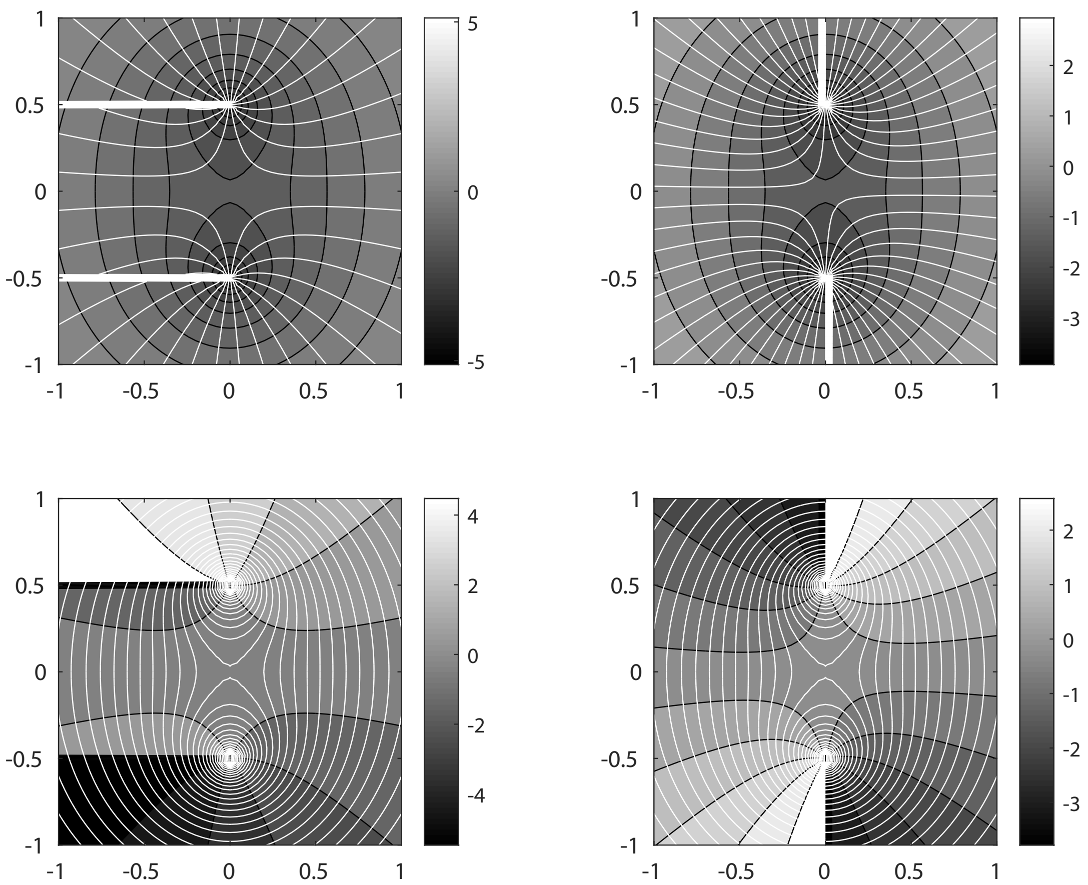

Figure 3. *Branch cut shift for superposition of two logarithms with same coefficient*

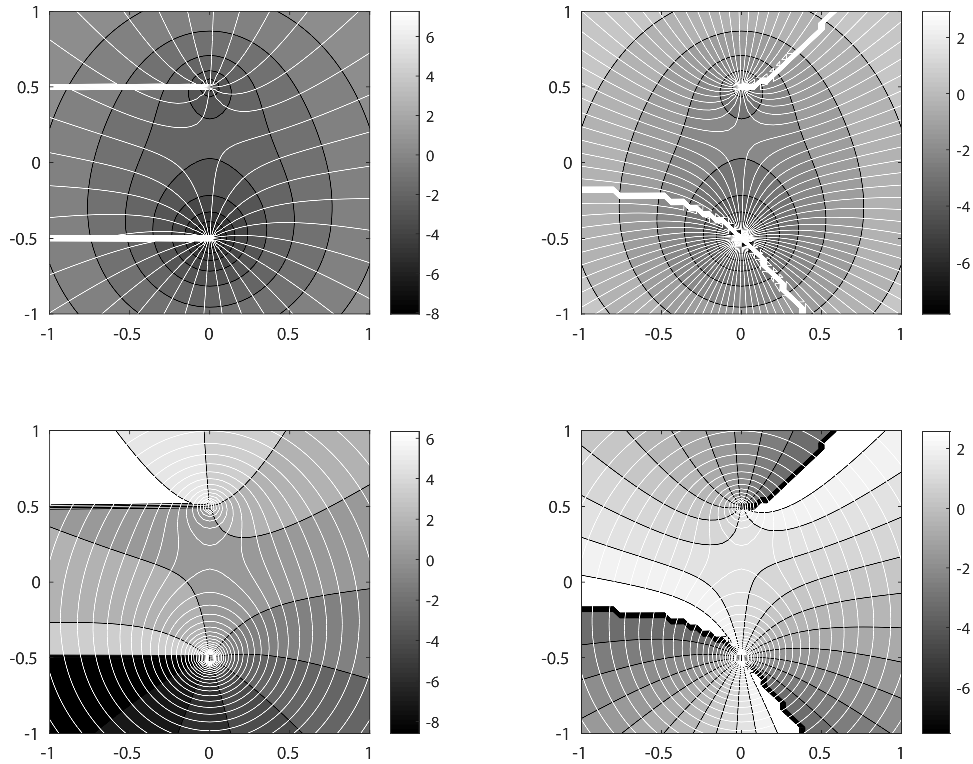

Figure 4. *Branch cut shift for superposition of logarithms with different coefficients*

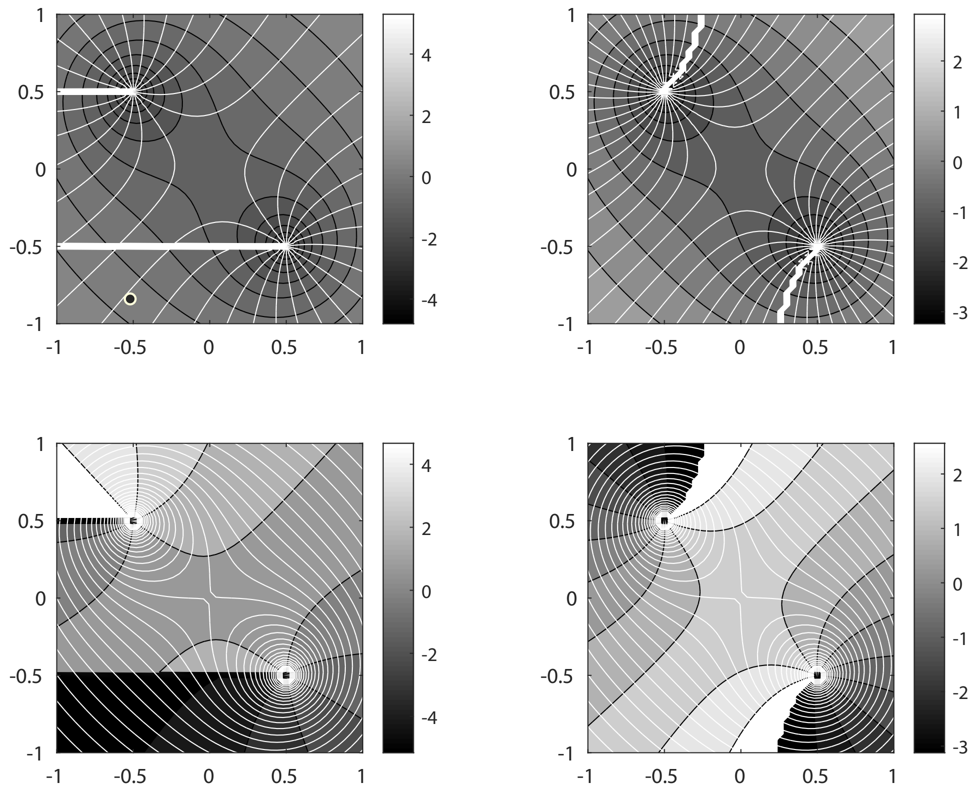

Figure 5. *Branch cut shift for superposition of logarithms with different positions*

streamfunction in the two lower sub-figures. What is most important additionally: the branch cuts for Ψ_1 and Ψ_2 are not at the same positions. While they extend from their base position horizontally to the left for Ψ_1, they extend vertically to the outer domain edges for Ψ_2.

In Figure 4 we use the same visualization for the superposition of the functions:

$$\Psi_1(z) = log(z-z_1) + 2log(z-z_2)$$
$$\Psi_2(z) = log\left[(z-z_1)(z-z_2)^2\right] \qquad (10)$$

Lets also check the visualization if we change the position of the logarithms, i.e. if we take $z_1 = \frac{1}{2}(i-1)$ and $z_2 = \frac{1}{2}(1-i)$ in equation (9). The result is shown in Figure 5 in the same manner as in the previous figures.

Figures 4 and 5 demonstrate that branch cuts change, if different formulations for the potential functions are used. For the basic superposition solution the cuts extend to negative infinity parallel to the real axis. For the functions Ψ_2, depicted in the sub-plots on the right of both figures, the branch cuts take a curved shape.

As a preliminary statement we conclude that the location of the branch cuts depends on the mathematical formulation of the function. The branch cuts may extend horizontally, vertically or take even a curved path within the domain. Using the genuine superposition, i.e. the addition of the log-terms, the branch cut locations are well- known: they extend from their base position in the domain parallel to the real axis towards negative infinity.

Domain Partition

The finding of the previous subchapter that the branch cuts of the superposed logarithmic potentials in the graphical visualisations extend horizontally towards the left margin can be utilized to construct a flow net without the inconsistencies for the streamlines. The proposed algorithm is outlined for the superposition of three logarithms:

$$\Psi(z) = \Psi_1 log(z-z_1) + \Psi_2 log(z-z_2) + \Psi_3 log(z-z_3) \qquad (11)$$

with $z_1 = 150(1+i)$: $z_2 = 180 + 250i$: $z_3 = 360 + 280i$ with coefficients. The entire domain is divided in bands and partitions as shown in Figure 6.

The algorithm starts with band 1, where the function is evaluated as given in equation (11) and visualised. Then one proceeds in the same way with block (2.1) in band 2, where again equation (11) is evaluated. What follows is block (2.2), where the jump at the branch cut for $\Psi_3 \log(z-z_3)$ has to be considered. The jump at the branch cut is $2\pi\Psi_3 i$. Thus one has to evaluate:

$$\Psi(z) = \Psi_1 log(z-z_1) + \Psi_2 log(z-z_2) + \Psi_3 log(z-z_3) + 2\pi\Psi_3 i \qquad (12)$$

The algorithm proceeds with the next band and partition on the left, i.e. block (3.1). Here equation (11) is evaluated. In the following block (3.2) one has to take the streamfunction jump at the branch for $\Psi_2 \log(z-z_2)$ into account and compute for the visualisation. Continuing in this manner, the function to evaluate in the next block (3.3) is:

$$\Psi(z) = \Psi_1 log(z-z_1) + \Psi_2 log(z-z_2) + \Psi_3 log(z-z_3) + 2\pi\Psi_2 i \qquad (13)$$

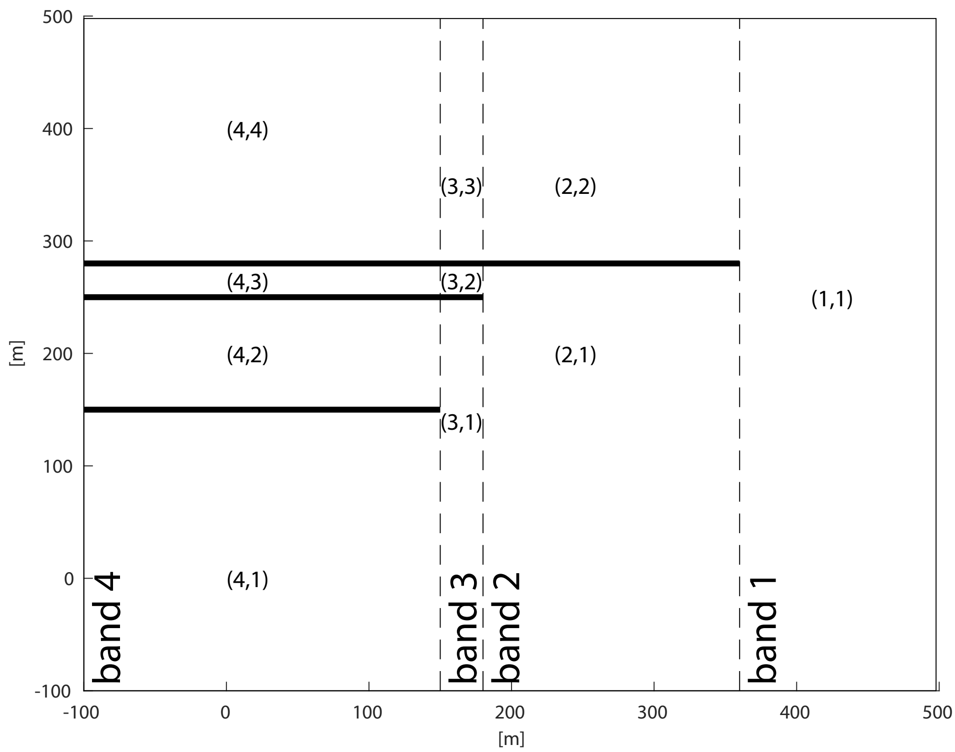

Figure 6: *Example of domain partitioning for 3-logarithm superposition*

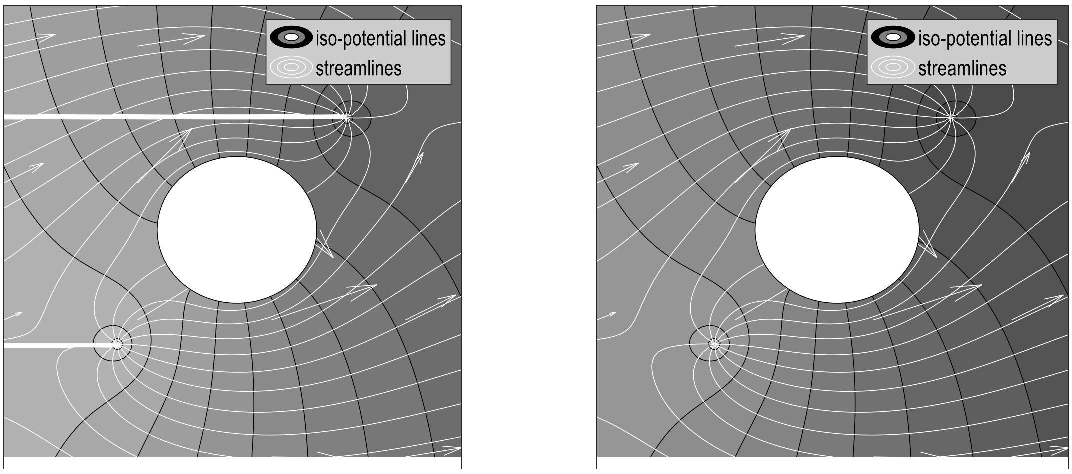

Figure 7. *Flow net with sink and source around a circular obstacle; without (left) and with (right) domain partition*

$$\Psi(z) = \Psi_1 log(z-z_1) + \Psi_2 log(z-z_2) + \Psi_3 log(z-z_3) + 2\pi(\Psi_2+\Psi_3)i \quad (14)$$

As final steps one has to proceed in the same way with the blocks (4.1) to (4.4) in band 4.

The described algorithm works for all superpositions of complex logarithms. If there are more then three logarithms to be included more bands and blocks have to be added, which are partitioned in the way outlined above.

Results

The described algorithm was implemented in MATLAB® and can be used for applications that include base flow and multiple line sinks and sources [8]. An example is shown in Figure 7.

Figure 7 visualises a flow field that includes a sink, a source and an isopotential circular obstacle. On the left the figure shows the result of a straightforward computation, while the on the right the result of the above described method with domain partition is depicted. Clearly the branch cuts that disturb the sub-figure on the right are eliminated in the plot on the right.

The flow fields, showing potentials in black and white and streamlines as white contour lines, are the same in both sub-plots. Where streamlines are dense, the velocity is high; where the streamlines are apart, the velocity is relatively low. Thus the streamline distribution clearly shows the high velocities in the close vicinity of the sink and the source.

As mentioned above streamfunction contouring, in contrast to particle tracing, allows visual balancing of flow rates by counting streamtubes between adjacent streamlines. Of the 12 streamtubes connected to the source on the lower left, 4 reach the sink at the upper right of the figures.

Conclusions

In the preceding we have shown how to visualize flow-nets of potential flow, avoiding the disturbances resulting from branch cuts of the principal branches of the potential functions. The basic recipe is to evaluate the functions in sub-domains instead of using a formula globally for the entire domain. The partition into sub-domains is determined by the branch cuts that depend on the formulation of the function. Thus it is important to select a formulation that allows an appropriate partition of the domain.

In several examples it was shown how the formulation of the potential determines the location of the branch cuts. Thus, in order to follow the outlined method a formulation has to be chosen that keeps the branch cuts simple and easy to access computationally. Only then the sub-domains can be identified.

For the complex logarithm it was demonstrated that the basic superposition formulation introduces a simple branch cut pattern: lines from a base point parallel to the real axis to negative infinity. This pattern allows the partitioning into rectangular blocks on which the potential formula is evaluated separately and sequentially, taking the jumps at the branch cuts into account.

Table 1. *List of complex functions and their branch cuts, following [5]*

Function	Branch Cut
√, zω, log	negative real axis
arcsin, arcos, arctanh	real axis without]-1,1[
arctan, arcsinh	imaginary axis without]-i,i[
arccosh	real axis ≤1

Similar partitioning can be performed for other functions as well. Table 1 lists functions that have branch cuts either along the real or imaginary axes. For these functions partitioning into rectangular blocks can be performed as demonstrated above.

The method outlined here overcomes the problems posed by branch cuts of functions of the complex plane to the complex plane. The author hopes it may help to utilize more the advantages of streamfunction contouring over other streamline tracing techniques.

Acknowledgements

The author is highly grateful to Kutaisi International University (KIU) in Georgia, where the topic of this paper was presented at the KIU Annual Conference on June 25th 2024, especially to Markus Neuhauser.

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